FINAL EXAM | CALCULUS I | Spring 2006

Make sure that you show enough work to justify your answers. Also please make sure that your work is written neatly and legibly.

NOTE: *To get full credit on this exam, you must complete each of the problems in section I. and any 10 of the problems in section II.*

Section I

1. (18) Find the derivative $\frac{dy}{dx}$ for each of the following:

   a) $y = 3x^5 \sin^{-1} x + 2\ln\sqrt{x^2+3x}$ b) $y = \cos(\tan(e^{4x}))$

2. (18) Find the limits:

   a) $\lim_{x\to 3} \frac{x^2+4x-21}{x-3}$ b) $\lim_{x\to 5+} \frac{x\sqrt{x-5}}{x+5}$

3. (18) Find antiderivatives for each of the following:

   a) $\frac{x^2}{(x^3+5)^2}$ b) $\frac{\cos\theta}{1+\sin\theta}$ c) $2xe^{x^2-1}$

Section II

4. (15) Use the limit definition of the derivative to determine, if possible, a value of k so that the function f given by

$$f(x) = \begin{cases} 2kx+1 & \text{if } x < 1 \\ x + 2kx^2 & \text{if } x > 1 \end{cases}$$

   is differentiable at $x = 1$. If not possible, write DNE. Give an explanation of how you arrived at your answer.

5. (15) Give an example of a function that is continuous at a point but not differentiable there. Briefly explain why your example works or give an explanation for why this can never happen.

6. (15) For the following function, draw a complete graph of the function, and indicate each of the following features: all intercepts, asymptotes, intervals where the function is increasing, where it is decreasing and the coordinates of any relative extrema. Also indicate intervals of upward and downward concavity.

$$F(x) = (x+2)^3(1-x)^2$$

7. (15) Sketch a graph of a function g which satisfies the following conditions:
   $g(-2) = 0$, $g(x) < 0$ for $x < -2$ and $g(x) > 0$ for $x > -2$;
   $g'(-2) = g'(3) = 0$;
   $g'(x) > 0$ for $x < -2$, $-2 < x < 3$;
   $g'(x) < 0$ for $x > 3$

8. (15) First find the third derivative of $y = \frac{1}{1-x}$. Then find an expression for the nth derivative.

9. (15) An object moves along a coordinate line so that its position p, in inches, at any time t, in seconds, is given by:

$$p = 2t^3 - 4t^2 + 5$$

Find the position of the object when its velocity is zero. Give a clear explanation for your answer.

10. (15)

$$\text{Let} \qquad G(x) = \frac{x^2 - 1}{x}$$

a) find the slope of the tangent line to the graph of G at the point (1,0).
b) find the equation of this tangent line.
c) find the local linear approximation to G at $x_o = 1$, and use it to estimate the value of $G(0.97)$.

11. (15) Find the derivative $\frac{dy}{dx}$ for each of the following:

a) $y = \dfrac{x^3 e^{-x}}{(4x^2 + 7)}$ b) $e^{xy} = x^3 + 3y^2$

12. (15) Find the limits:

a) $\displaystyle\lim_{x\to\infty} \frac{5x^2 - 3x + 43}{52x - 3x^2}$ b) $\displaystyle\lim_{x\to 0} \frac{e^x - e^{-x}}{4 \sin x}$

13. (15) For each of the following statements: if the statement is true, carefully explain why it is true; if the statement is false, then give a counterexample.
a) If the function F is differentiable at a point a, then it is continuous at a.
b) If G is a function such that $G'(c) = 0$, then c is point at which G has a relative maximum or a relative minimum.
c) If F is a function whose domain is the interval $[a, b]$ and $F'(x) = 0$ for every x in $[a, b]$, then F is constant on its domain.

14. (15) Evaluate the definite integrals.

a) $\displaystyle\int_7^{10} \frac{dx}{\sqrt{x - 6}}$ b) $\displaystyle\int_1^3 \frac{\ln x}{x}\, dx$

15. (15) A rectangular playground is to be fenced off and divided in two by another fence parallel to one side of the playground. Six hundred feet of fencing is used. Find the dimensions of the playground that will enclose the greatest total area.