Final Examination
Calculus I (MATH-156), Spring 2007

Max Points 200

Answer to all questions on your examination booklet. Provide details where intermediate steps are obviously needed. Adjust your strategy to the given marking scheme.

I. *(10 points - 1 pt each)* Answer true or false

a) If $|x|<\epsilon$ for every positive number ϵ then $x=0$.
b) The equation of any line can be written in point-slope form.
c) The graph of $y = f(x-c)$ shifts the graph of $y = f(x)$ to the left c units.
d) $\log(x+y)=\log(x)+\log(y)$.
e) The natural domain of $f(x) = \sqrt{\frac{x}{4-x}}$ is the interval [0,4).
f) (fog)(x) means the same thing as g(f(x)).
g) $|x-2|$ equals x-2 when x is positive and 2-x when x is strictly negative.
h) If $p(x)$ is a polynomial, then $\lim_{x\to a} p(x) = p(a)$.
i) If $f(c)$ is not defined, then $\lim_{x\to c} f(x)$ does not exist.
j) An even function plus an odd function is an odd function.

II. *(35 points - 5 pts each)* Compute the following limits

a) $\lim_{x\to -1} \frac{2x^2}{x^8-2x^2-x}$ **b)** $\lim_{x\to 2} \frac{x^3+x^2-4x-4}{x-2}$ **c)** $\lim_{x\to\infty} \frac{1-\frac{2}{x}}{x^2-4}$ **d)** $\lim_{x\to 1^-} \frac{\sqrt{1+x}}{4+4x}$

e) $\lim_{x\to 1} \frac{x-4}{\sqrt{x}-2}$ **f)** $\lim_{x\to\infty} \frac{\sqrt{3x+4}-2}{x}$ **g)** $\lim_{x\to\infty} \frac{4\sqrt{x}+3}{x+7}$

III. *(20 points)* The equation for the volume of a sphere is $V = \frac{4}{3}\pi r^3$.
a) A spherical balloon is expanding from the sun's heat. Find the rate of change of its volume with respect to its radius when the radius is 5 meters. (5 pts)
b) Use differentials to approximate the change in volume of the balloon when its radius increases from 5 to 5.1 meters. (5 pts)
c) If the volume of the balloon is increasing at a constant rate of 10 cubic meters per hour, how fast is its radius increasing when the radius is 5 meters? (10 pts)

IV. *(35 points - 5 pts each)* Compute the derivatives of the following functions
a) $f(x) = \log(\ln x)$ **b)** $g(x) = e^x \sin^4(x^2+3)$ **c)** $h(x) = \frac{2x-1}{x-1}$ **d)** $k(x) = \int_0^x \arctan(t)\,dt$
e) $l(x) = \arctan(x)$ **f)** $m(x) = e^{(x-e^{x^2})}$ **g)** $n(x) = \int_4^{2x} \ln(t)\,dt$

V. *(20 points - 10 pts each)* A manufacturer of tin cans is to make a cylindrical can holding 80 cubic inches. Recall that the volume of a cylinder is $V = \pi r^2 h$.
a) What is the least amount of tin needed to make the can?
b) What is the radius and height of the can requiring the least amount of tin?

VI. *(20 points - 5 pts each)* **Evaluate the following Indefinite Integrals**

a) $\int (3\sin(x) - x)\,dx$ **b)** $\int \cos(x)\cos(\sin(x))\,dx$ **c)** $\int (1+\sin t)^9 \cos t\,dt$ **d)** $\int \frac{x^2}{1+x^6}\,dx$

VII. *(15 points - 5 pts each)* **Evaluate the following definite Integrals**

a) $\int_0^1 e^{0.01x}\,dx$ **b)** $\int_1^3 \frac{1+2x-x^2}{x^3}\,dx$ **c)** $\int_0^1 \left(x^3 - 3x^2 + 3\sqrt{x}\right)dx$

VIII. *(45 points)* **Let** $G(x) = (x+2)(x-1)^2$.

a) Find its x and y intercepts. (5 pts)

b) Find the intervals where G **is increasing or decreasing. Find the stationary points and local extrema. (10 pts)**

c) Find the intervals where G **is concave up or down. Find the inflection points if any. (10 pts)**

d) Sketch the graph of G **representing clearly the information gathered above. (10 pts)**

e) Evaluate $\int_{-2}^{1} G(x)\,dx$. **Represent the corresponding area on the graph of** G. **(10 pts)**

Calculus I Final Exam Solutions Spring 07

I.

a) T
b) F
c) F
d) F
e) T
f) F
g) F
h) T
i) F
j) F

II

a) $\lim_{x\to -1} \frac{2x}{x^8-2x^2-x} = \lim_{x\to -1} \frac{2}{x^7-2x-1}$ dine

b) $\lim_{x\to 2} \frac{x^3+x^2-4x-4}{x-2} = \lim_{x\to 2} \frac{x^2(x+1)-4(x+1)}{x-2}$

$= \lim_{x\to 2} \frac{(x^2-4)(x+1)}{x-2} = \lim_{x\to 2} \frac{(x+2)(x-2)(x+1)}{x-2} = \lim_{x\to 2} (x+2)(x+1)$

$= 12$

OR

$\lim_{x\to 2} \frac{x^3+x^2-4x-4}{x-2} = \lim_{x\to 2} \frac{3x^2+2x-4}{1}$ by L'Hopital's Rule

$= 12+4-4 = 12$

c) $\lim_{x\to\infty} \frac{1-\frac{2}{x}}{x^2-4} = \lim_{x\to\infty} \frac{\left(\frac{x-2}{x}\right)}{x^2-4} = \lim_{x\to\infty} \frac{x-2}{x(x^2-4)}$

$= \lim_{x\to\infty} \frac{1}{x(x+2)} = 0$

d) $\lim_{x\to 1^-} \frac{\sqrt{1+x}}{4+4x} = \frac{\sqrt{2}}{8}$

e) $\lim_{x\to 1} \frac{x-4}{\sqrt{x}-2} = \frac{-3}{-1} = 3$

f) $\lim_{x\to\infty} \frac{\sqrt{3x+4}-2}{x} = \lim_{x\to\infty} \left(\frac{\sqrt{3x-4}}{x} - \frac{2}{x}\right)$

$\sqrt{\lim_{x\to\infty} \frac{3x+4}{x^2}} - \lim_{x\to\infty} \frac{2}{x} = 0-0 = 0$

g) $\lim_{x\to\infty} \frac{4\sqrt{x}+3}{x+7} = \lim_{u\to\infty} \frac{4u+3}{u^2+7}$ where $u=\sqrt{x}$

$= 0$

III

a) $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dr}(5) = 4\pi r^2 \Big|_{r=5} = 100\pi$

b) $dV = 4\pi r^2 \, dr$

$= 4\pi (5)^2 (.1) = 10\pi$ meters

c) $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \Rightarrow 10 = 4\pi (5)^2 \frac{dr}{dt}$

So $\frac{dr}{dt} = \frac{10}{100\pi} = \frac{1}{10\pi}$ meters/sec

IV

a) $\left[\log_{10}(\ln x)\right]' = \left(\frac{\ln(\ln x)}{\ln 10}\right)' = \frac{1}{\ln 10}\left(\frac{1}{\ln x}\right)(\ln x)'$

$= \frac{1}{(x \ln x)\ln 10}$

b) $g(x) = e^x \sin^4(x^2+3)$

$g'(x) = e^x \sin^4(x^2+3) + e^x \, 4\sin^3(x^2+3)\cos(x^2+3)(2x)$

$= e^x \sin^3(x^2+3)\left[\sin(x^2+3) + 8x\cos(x^2+3)\right]$

c) $h(x) = \frac{2x-1}{x-1}$

$h'(x) = \frac{(2x-1)'(x-1) - (x-1)'(2x-1)}{(x-1)^2} = \frac{2(x-1) - (2x-1)}{(x-1)^2}$

$= \frac{-1}{(x-1)^2}$

d) $k(x) = \int_0^x \tan^{-1} t \, dt \Rightarrow k'(x) = \tan^{-1} x$

e) $l(x) = \tan^{-1} x \Rightarrow l'(x) = \frac{1}{x^2+1}$

f) $m(x) = e^{x-e^{x^2}} \Rightarrow m'(x) = e^{x-e^{x^2}}\left(x - e^{x^2}\right)' = e^{x-e^{x^2}}\left(1 - e^{x^2}(x^2)'\right)$

$= \left(1 - 2xe^{x^2}\right)\exp\left(x - e^{x^2}\right)$

g) $n(x) = \int_4^{2x} \ln t \, dt \Rightarrow n'(x) = (\ln 2x)(2x)' = 2\ln 2x$

V

a) $80 = \pi r^2 h$, so $h = \frac{80}{\pi r^2}$

$A = 2\pi r^2 + 2\pi r h$, where A= surface area of can

Area of top + bottom

Area of side

$\therefore A(r) = 2\pi r^2 + 2\pi r\left(\frac{80}{\pi r^2}\right)$

$= 2\pi r^2 + \frac{160}{r}$

$A'(r) = 4\pi r - \frac{160}{r^2}$, so $A'(r)=0 \Rightarrow r = \left(\frac{40}{\pi}\right)^{1/3} = 2\sqrt[3]{\frac{5}{\pi}}$

So $h = \left(\frac{80}{\pi}\right)\left(\frac{1}{2}\right)^2\left(\frac{\pi}{5}\right)^{2/3} = \frac{20}{(25\pi)^{1/3}}$

of course, r + h, can take many equivalent forms,

or $h = \left(\frac{80}{\pi}\right)\left(\frac{\pi}{40}\right)^{2/3} = 2\left(\frac{40}{\pi}\right)\left(\frac{\pi}{40}\right)^{2/3} = 2\left(\frac{40}{\pi}\right)^{1/3}$

So minimum area for volume of 80 in³ is

$2\pi\left(\frac{40}{\pi}\right)^{2/3} + 160\left(\frac{\pi}{40}\right)^{1/3}$

$= 80\left(\frac{\pi}{40}\right)^{1/3} + 160\left(\frac{\pi}{40}\right)^{1/3} = 240\left(\frac{\pi}{40}\right)^{1/3}$ sq inches

VI

a) $\int 3\sin x - x\,dx = 3\int \sin x\,dx - \int x\,dx = -3\cos x - \frac{x^2}{2} + C$

b) $\int \cos x \cos(\sin x)\,dx = \int \cos u\,du$, where $u = \sin x$, $du = \cos x\,dx$

$= \sin u + C = \sin(\sin x) + C$

c) $\int (1+\sin t)^9 \cos t\,dt = \int u^9\,du$, where $u = 1+\sin t$, $du = \cos t\,dt$

$= \frac{u^{10}}{10} + C = \frac{(1+\sin t)^{10}}{10} + C$

d) $\int \frac{x^2}{1+x^6}\,dx = \int \frac{\frac{1}{3}du}{1+u^2}$ where $u = x^3$, $du = 3x^2dx$

$= \frac{1}{3}\arctan u + C = \frac{1}{3}\arctan x^3 + C$

VIII $G(x) = (x+2)(x-1)^2$

a) y-intercept: $y = G(0) = (2)(-1)^2 = 2$

x-intercept: $G(x) = 0 \Rightarrow x = 1, -2$

b) $G'(x) = (x+2)'(x-1)^2 + (x+2)\left[(x-1)^2\right]'$

$= (x-1)^2 + 2(x+2)(x-1)$

$= (x-1)\left[(x-1) + 2(x+2)\right] = (x-1)(3x+3)$

$= 3(x-1)(x+1)$

$G'(x) = 0 \Rightarrow x = \pm 1$

rel. max rel. min

$+ + + + 0 - - - - - - 0 + + + +$ ← sign of G'

$-1 \qquad 1$

From analysis above,

the stationary points are $x = \pm 1$

By 1st deriv. test, $G(x)$ has a relative min of 0 at $x = 1$

and a relative max of 4 at $x = -1$

G is increasing for $|x| > 1$

G is decreasing on $|x| < 1$

c) $G'(x) = 3(x^2-1) \Rightarrow G''(x) = 6x$

$G''(x) = 0 \Rightarrow x = 0$ $\qquad - - - 0 + + + +$ ← sign of G''

0

So $(0, 2)$ is an inflection pt.; G is concave up for $x > 0$

G is concave down for $x < 0$

VII

a) $\int_0^1 e^{.01x}\,dx = \frac{e^{.01x}}{.01}\Big|_0^1 = 100(e^{.01} - 1)$

b) $\int_1^3 \frac{1+2x-x^2}{x^3}\,dx = \int_1^3 x^{-3} + 2x^{-2} - x^{-1}\,dx$

$= \left(\frac{x^{-2}}{-2} + \frac{2x^{-1}}{-1} - \ln x\right)\Big|_1^3$

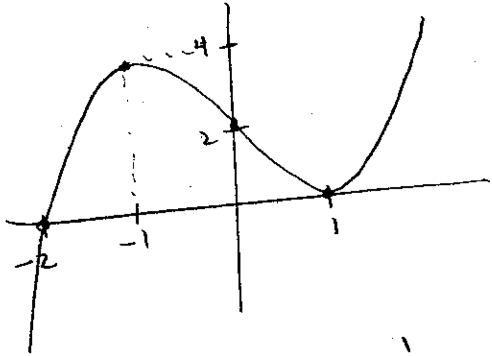
$= \left(\frac{-1}{2(3)^2} - \frac{2}{3} - \ln 3\right) - \left(-\frac{1}{2} - 2 - \ln 1\right)$

$= -\frac{1}{18} - \frac{2}{3} - \ln 3 + \frac{1}{2} + 2 + 0 = \frac{16}{9} - \ln 3$

c) $\int_0^1 x^3 - 3x^2 + 3\sqrt{x}\,dx$

$= \left(\frac{x^4}{4} - x^3 + 2x^{3/2}\right)\Big|_0^1 = \frac{1}{4} - 1 + 2 = \frac{5}{4}$

d)



e) $\displaystyle\int_{-2}^{1} G(x)\,dx = \int_{-2}^{1} (x+2)(x-1)^2\,dx = \int_{-2}^{1} (x-1+3)(x-1)^2\,dx$

$$\left(\begin{array}{l} \text{Let } u = x-1 \\ \quad du = dx \\ x=-2 \Leftrightarrow u=-3 \\ x=1 \Leftrightarrow u=0 \end{array}\right)$$

$$= \int_{-3}^{0} (u+3)u^2\,du$$

$$= \int_{-3}^{0} u^3 + 3u^2\,du$$

$$= \left(\frac{u^4}{4} + u^3\right)\Big|_{-3}^{0}$$

$$= -\frac{(-3)^4}{4} - (-3)^3$$

$$= -\frac{81}{4} + 27$$

$$= \frac{27}{4}$$