Final Examination
Calculus I (MATH-156), Spring 2007

Max Points 200

Answer to all questions on your examination booklet. Provide details where intermediate steps are obviously needed. Adjust your strategy to the given marking scheme.

I. (10 points - 1 pt each) Answer true or false
a) If |x|<ε for every positive number ε then x=0.
b) The equation of any line can be written in point-slope form.
c) The graph of y = f(x-c) shifts the graph of y = f(x) to the left c units.
d) log(x+y)=log(x)+log(y).
e) The natural domain of f(x) = \sqrt{\frac{x^2}{x^3}} is the interval [0,4).
f) (fog)(x) means the same thing as g(f(x)).
g) |x-2| equals x-2 when x is positive and 2-x when x is strictly negative.
h) If p(x) is a polynomial, then lim_{x->a} p(x) = p(a).
i) If f(c) is not defined, then lim_{x->c} f(x) does not exist.
j) An even function plus an odd function is an odd function.

II. (35 points - 5 pts each) Compute the following limits
a) \lim_{x\to-1} \frac{2x^2}{x^8-2} x^2 - x
b) \lim_{x\to2} \frac{x^3+x^2-4x-4}{x-2}
c) \lim_{x\to\infty} \frac{1-x}{x^2-4}
d) \lim_{x\to1} \frac{\sqrt{1+x}}{4+4x}

e) \lim_{x\to1} \frac{x-4}{\sqrt{x^2-2}}
f) \lim_{x\to\infty} \frac{\sqrt{3x+4}-2}{x}
g) \lim_{x\to\infty} \frac{4\sqrt{x}+3}{x+7}

III. (20 points) The equation for the volume of a sphere is V = \frac{4}{3} \pi r^3.
a) A spherical balloon is expanding from the sun's heat. Find the rate of change of its volume with respect to its radius when the radius is 5 meters. (5 pts)
b) Use differentials to approximate the change in volume of the balloon when its radius increases from 5 to 5.1 meters. (5 pts)
c) If the volume of the balloon is increasing at a constant rate of 10 cubic meters per hour, how fast is its radius increasing when the radius is 5 meters? (10 pts)

IV. (35 points - 5 pts each) Compute the derivatives of the following functions
a) f(x) = log(lnx)  b) g(x) = e^x \sin^4(x^2 + 3)  c) h(x) = \frac{2x-1}{x-1}  d) k(x) = \int_0^x \arctan(t) \, dt
e) l(x) = \arctan(x)  f) m(x) = e^{e^{-x^2}}  g) n(x) = \int_0^x \ln(t) \, dt

V. (20 points - 10 pts each) A manufacturer of tin cans is to make a cylindrical can holding 80 cubic inches. Recall that the volume of a cylinder is V = \pi r^2 h.
a) What is the least amount of tin needed to make the can?
b) What is the radius and height of the can requiring the least amount of tin?
VI. (20 points - 5 pts each) Evaluate the following Indefinite Integrals
a) \( \int (3 \sin(x) - x) \, dx \)  b) \( \int \cos(x) \cos(\sin(x)) \, dx \)  c) \( \int (1 + \sin(t))^9 \cos t \, dt \)  d) \( \int \frac{x^3}{1+x^4} \, dx \)

VII. (15 points - 5 pts each) Evaluate the following definite Integrals
a) \( \int_0^1 e^{0.01x} \, dx \)  b) \( \int_1^3 \frac{1+2x-x^2}{x} \, dx \)  c) \( \int_1^3 (x^3 - 3x^2 + 3 \sqrt{x}) \, dx \)

VIII. (45 points) Let \( G(x) = (x + 2)(x - 1)^2 \).

a) Find its x and y intercepts. (5 pts)
b) Find the intervals where \( G \) is increasing or decreasing. Find the stationary points and local extrema. (10 pts)
c) Find the intervals where \( G \) is concave up or down. Find the inflection points if any. (10 pts)
d) Sketch the graph of \( G \) representing clearly the information gathered above. (10 pts)
e) Evaluate \( \int_1^3 G(x) \, dx \). Represent the corresponding area on the graph of \( G \). (10 pts)
I.

9) T

10) F

11) F

12) F

13) T

II.

a) \( \lim_{{x \to 1}} \frac{2x}{{x^2 - 2x - 1}} = \frac{2}{{-1}} = \text{DNE} \)

b) \( \lim_{{x \to 2}} \frac{x^3 + x^2 - 4x - 4}{x - 2} = \lim_{{x \to 2}} \frac{x(x+1)-4(x+1)}{x-2} = \lim_{{x \to 2}} \left( \frac{x+1}{x-2} \right) = 3 \)

\( \lim_{{x \to \infty}} \frac{3x^2 + 4}{x^2} = \lim_{{x \to \infty}} \frac{3 + \frac{4}{x^2}}{1} = 3 \)

\( \lim_{{x \to \infty}} \frac{4x^3 + 3}{x^4} = \lim_{{u \to \infty}} \frac{4u^3 + 3}{u^4} \)

where \( u = \sqrt{x} \)

\( \lim_{{x \to 1}} \frac{\sqrt{1+x}}{4+4x} = \frac{\sqrt{5}}{8} \)

\( \lim_{{x \to 1}} \frac{x-4}{x^2 - 2} = \frac{-3}{1} = 3 \)

\( \lim_{{x \to \infty}} \frac{\sqrt{3x^4 + 2}}{x} = \lim_{{x \to \infty}} \left( \frac{\sqrt{3x^4 + 2}}{x} \right) \)
\[
\frac{4}{3} \pi r^3 \Rightarrow \frac{dV}{dr} = 4 \pi r^2 \bigg|_{r = 5} = 100 \pi \\
\]

\[dV = 4 \pi r^2 \, dr = 4 \pi (5)^2 \, (1) = 100 \pi \text{ meters}^3\]

\[
dV = 4 \pi r^2 \, dr \Rightarrow 10 = 4 \pi (5)^2 \, \frac{dr}{dt} \]

\[\text{So, } \frac{dr}{dt} = \frac{1}{10 \pi} \text{ meters per second} \]

\[
\left[ \log_{10} (\ln x) \right]' = \left( \frac{\ln(\ln x)}{\ln 10} \right)' = \frac{1}{\ln 10} \left( \frac{1}{\ln x} \right) (\ln x)' \\
= \frac{1}{(x \ln x) \ln 10}
\]

\[
g(x) = e^x \sin^4(x^2 + 3) \\
g'(x) = e^x \sin^4(x^2 + 3) + e^x 4 \sin^3(x^2 + 3) \cos(x^2 + 3) (2x) \\
= e^x \sin^3(x^2 + 3) \left[ \sin(x^2 + 3) + 8x \cos(x^2 + 3) \right]
\]

\[
h(x) = \frac{2x - 1}{x - 1} \\
h'(x) = \left( \frac{2x - 1}{x - 1} \right)' - \left( \frac{2x - 1}{x - 1} \right)'' = \frac{-1}{(x - 1)^2}
\]

\[
k(x) = \int_0^x \tan^{-1} t \, dt \Rightarrow k'(x) = \tan^{-1} x
\]

\[
l(x) = \tan^{-1} x \Rightarrow l'(x) = \frac{1}{x^2 + 1}
\]

\[
m(x) = e^{x-e^x} \Rightarrow m'(x) = e^{x-e^x}(x-e^x)' = e^{x-e^x}(1-e^x)(1-e^x') \\
= (1 - 2xe^{x^2}) e^{x-e^x}
\]

\[
n(x) = \int_4^{2x} \ln t \, dt \Rightarrow n'(x) = (\ln 2x)(2x)' = 2 \ln 2x
\]
V

9) \(80 = \pi^2 h\), \(\therefore h = \frac{80}{\pi^2}\)

\[A = 2\pi r^2 + 2\pi rh\], where
\(A = \text{surface area of cone}\)

\[\text{Area of top} + \text{Area of side}\]

\[\therefore A(r) = 2\pi r^2 + 2\pi \left(\frac{80}{\pi^2}\right)\]
\[= 2\pi r^2 + \frac{160}{r}\]

\[A'(r) = 4\pi r - \frac{160}{r^2}\], so \(A'(r) = 0\)
\[r = \left(\frac{40}{\pi}\right)^{\frac{1}{3}}\]

\[\text{Of course, } r + h\]
\[\text{can take many equivalent forms}\]

\[\therefore h = \left(\frac{80}{\pi}\right)\left(\frac{\pi}{4}\right)^{\frac{1}{3}} = 2\left(\frac{40}{\pi}\right)^{\frac{1}{3}}\]

\[= \left(\frac{40}{\pi}\right)^{\frac{1}{3}}\]

So minimum area for volume of 80 m³ is
\[2\pi \left(\frac{40}{\pi}\right)^{\frac{1}{3}} + 160 \left(\frac{\pi}{40}\right)^{\frac{1}{3}}\]
\[= 80 \left(\frac{\pi}{40}\right)^{\frac{1}{3}} + 160 \left(\frac{\pi}{40}\right)^{\frac{1}{3}} = 240 \left(\frac{\pi}{40}\right)^{\frac{1}{3}} \text{ square meters}\]
\[ G(x) = (x+2)(x-1)^2 \]

a) \( y\)-intercept: \( y = G(0) = 2(0)(-1)^2 = 2 \)

b) \( G(x) = (x+2)(x-1)^2 + (x+2)(x-1)^2 \)

\[ = x(x-1)^2 + 2(x+2)(x-1) \]

\[ = (x-1)[x(x-1)+2(x+2)] = (x-1)(3x+3) \]

\[ = 3(x-1)(x+1) \]

\[ G'(x) = 0 \Rightarrow x = \pm 1 \]

From analysis above:

the stationary points are \( x = \pm 1 \).

By 1st derivative test, \( G(x) \) has a relative min at \( x = 1 \) and a relative max at \( x = -1 \).

\( G \) is increasing for \( x > 1 \)

\( G \) is decreasing on \( x < -1 \).

c) \( G(x) = 3(x-1) \Rightarrow G''(x) = 6x \)

\[ G''(x) = 0 \Rightarrow x = 0 \]

So, \( (0, 2) \) is an inflection pt; \( G \) is concave up for \( x > 0 \) and concave down for \( x < 0 \).

a) \[ \int_0^1 e^{-0.1x} \, dx = \left. \frac{e^{-0.1x}}{-0.1} \right|_0^1 = 100(e^{-0.1} - 1) \]

b) \[ \int_1^3 \frac{1 + 2x - x^2}{x^3} \, dx = \int_1^3 x^{-3} + 2x^{-2} - x^{-1} \, dx \]

\[ = \left. \left( \frac{x^{-2}}{-2} + \frac{2x^{-1}}{-1} - \ln x \right) \right|_1^3 \]

\[ = \left( \frac{1}{2} + \frac{2}{3} - \ln 3 \right) - \left( -\frac{1}{2} - 2 - \ln 1 \right) \]

\[ = \frac{1}{6} - \frac{2}{3} - \ln 3 + \frac{1}{2} + 2 + 0 = \frac{5}{6} - \ln 3 \]

c) \[ \int_0^2 x^3 - 3x^2 + 3x - 1 \, dx \]

\[ = \left. \left( \frac{x^4}{4} - x^3 + 2x^2 \right) \right|_0^2 = \frac{4}{4} - 1 + 2 = \frac{5}{4} \]
\begin{align*}
\int_{-2}^{1} G(x) \, dx &= \int_{-2}^{1} (x+2)(x-1)^2 \, dx \\
&= \int_{-2}^{1} (x-1+3)(x-1)^2 \, dx \\
&= \int_{-3}^{0} (u+3)u^2 \, du \\
&= \int_{-3}^{0} u^3 + 3u^2 \, du \\
&= \left( \frac{u^4}{4} + u^3 \right) \bigg|_{-3}^{0} \\
&= -\frac{(-3)^4}{4} - (-3)^3 \\
&= -\frac{81}{4} + 27 \\
&= \frac{27}{4}
\end{align*}