Calculus II - Final Examination - Fall 2005

December 13, 4-6 PM

no calculators

Please check your answers and explain your reasoning.

The questions with a star are worth 20 points, the others are 10 points.

1. Find the sum of the series

$$2 - \frac{8}{3!} + \frac{32}{5!} - \frac{128}{7!} + \cdots.$$

2. * Let $f(x) = \frac{1}{3-x}$. Find the Taylor-MacLaurin series at $x = 0$ for $f(x)$ and find the interval where the series converges.

3. Determine if the sequence

$$\left\{\frac{n}{3n+1}\right\}_{n=0}^{\infty}$$ is strictly increasing, strictly decreasing, both, or neither.

If the sequence converges, find the limit.

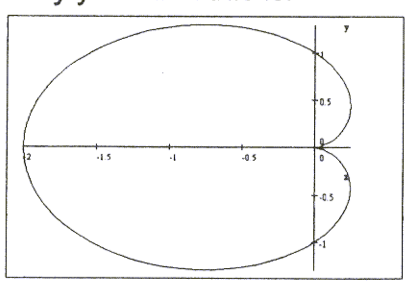
4. * Does the series

$$\sum_{k=1}^{\infty} \frac{k^2}{3^k}$$

converge or diverge? Why? If it converges is the convergence absolute or conditional?

5. Convert the curve $r = 2\sin\theta + 2\cos\theta$ into x,y coordinates and identify the type of curve. Then graph the curve.

6. * Consider the area enclosed by the cardiod $r = 1 - \cos\theta$. Without calculating which of the following answer(s) might be reasonable? Why? –498, 4, .0092, 6, 18, 498. Finsh by computing the actual area. If your answer does not seem reasonable please explain why you think this is.



7. * Evaluate the following integrals:

a. $\int \frac{\ln x}{x^2} dx$

b. $\int_1^{\infty} \frac{\ln x}{x^2} dx$

c. Why would a negative answer for 7.b make you suspicious?

8. * Find the following integrals

a. $\int \frac{du}{u^2-6u+5}$

b. $\int xe^{-2x}dx$

c. $\int \tan^6 z \sec^2 z\, dz$

9. * Now try

a. $\int \frac{dx}{\sqrt{x}(x+4)}$ Hint: $u^2 = x$

b. $\int_{12}^{\infty} \frac{dx}{\sqrt{x}(x+4)}$

10. * Let R be the region enclosed by the equations $y = x - 1$ and $x = 3 - y^2$.

a. Sketch the region R and find the points of intersection of the two curves.

b. Just looking at your sketch give a range of values for the area.

c. Find the area enclosed by R.

11. Find the average value of $f(x) = 2 + |x|$ on the interval $[-1, 3]$.

12. * Find the volume generated when the area under one arch of the curve $y = \sin x$ is revolved about the x-axis.

13. For each of the following statements, indicate whether it is True or False. For these problems a T or F suffices but a point will deducted for each wrong answer.

a. If $\lim_{n\to\infty} a_n = L$ then $\lim_{n\to\infty} a_{n+3} = L$

b. If $\lim_{n\to\infty} a_{2n} = L$ then $\lim_{n\to\infty} a_n = L$

c. If $\sum_{n=0}^{\infty} a_n x^n$ converges at $x = 2$, then it also converges at $x = -2$

d. If $\sum_{n=0}^{\infty} a_n x^n$ converges at $x = 2$, then it also converges at $x = -1$

e. If $\sum_{n=10,000}^{\infty} a_n$ converges, then so does $\sum_{n=1}^{\infty} a_n$

f. If $0 \le a_n \le b_n$ for all n, and if $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ also diverges.

g. $\int_0^1 \frac{\cos x}{x}$ is an improper integral.

h. The graph of $r = \cos 4\theta$ is a rose with 4 petals.

i. The graph of $r = 2\cos\theta$ is a circle.

j. The Maclaurin polynomial of order 3 for $f(x) = 2x^3 - x^2 + 7x - 11$ is the same as $f(x)$.

Calc 2 Fall 05 Solutions

#1 $2 - \frac{8}{3!} + \frac{32}{5!} - \frac{128}{7!} + \dots$

$= \frac{2}{1!} - \frac{2^3}{3!} + \frac{2^5}{5!} - \frac{2^7}{7!} + \dots$

$= \sin(2)$

#2 $f(x) = \frac{1}{3-x} = \frac{1}{3(1-\frac{x}{3})} = \frac{1}{3} \cdot \frac{1}{1-(\frac{x}{3})}$

$= \frac{1}{3} \sum_{k=0}^{\infty} \left(\frac{x}{3}\right)^k = \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^{k+1} x^k$

This Geometric Series converge for $\left|\frac{x}{3}\right| < 1$, or $|x| < 3$

$\therefore$ the interval of convergence is $(-3, 3)$

#3 $\lim_{n\to\infty} \frac{n}{3n+1} = \lim_{n\to\infty} \frac{1}{3+\frac{1}{n}} = \frac{1}{3}$

Consider. $f(x) = \frac{x}{3x+1}$.

$f'(x) = \frac{(x)'(3x+1) - x(3x+1)'}{(3x+1)^2} = \frac{(3x+1) - 3x}{(3x+1)^2} = \frac{1}{(3x+1)^2} > 0$

$\therefore$ $f(n) = \frac{n}{3n+1}$ is an increasing sequence.

Alternative method:

$\frac{a_{n+1}}{a_n} = \left(\frac{n+1}{3(n+1)+1}\right) \Big/ \left(\frac{n}{3n+1}\right) = \frac{(n+1)(3n+1)}{(3n+4)(n)} = \frac{3n^2+4n+1}{3n^2+4n} > 1$

$\therefore a_{n+1} > a_n$ for all n

#4 Let $a_k = \frac{k^2}{3^k}$.

Then $\frac{a_{k+1}}{a_k} = \frac{(k+1)^2}{3^{k+1}} \cdot \frac{3^k}{k^2} = \frac{1}{3}\left(\frac{k+1}{k}\right)^2 \to \frac{1}{3}$ as $k \to \infty$

$\therefore$ by the Ratio Test, $\sum \frac{k^2}{3^k}$ converges absolutely

#5 $r = 2\sin\theta + 2\cos\theta$

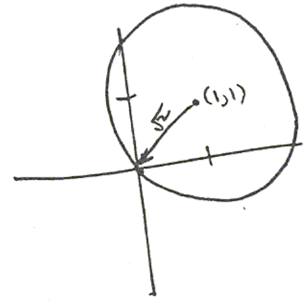
$r^2 = 2r\sin\theta + 2r\cos\theta$

$x^2 + y^2 = 2y + 2x$

$x^2 - 2x + y^2 - 2y = 0$

$x^2 - 2x + 1 + y^2 - 2y + 1 = 1 + 1$

$(x-1)^2 + (y-1)^2 = 2$ ← Circle centered at $(1,1)$ of radius $\sqrt{2}$



#6 From the plot, it appears that the cardiod fits inside of a rectangle of width 2.5 and height 3.
Also, it appears that a rectangle approximately of dimensions width × height = 1.5 × 2 will ~~fit~~ fit inside of the cardiod.
So its area is between 3 and 7.5.
Therefore, the only reasonable estimates on the given list are 4 or 6.

$$\text{Area} = 2\int_0^{\pi} \tfrac{1}{2} r^2(\theta)\,d\theta = \int_0^{\pi} r^2\,d\theta = \int_0^{\pi} (1-\cos\theta)^2\,d\theta$$

$$= \int_0^{\pi} (1 - 2\cos\theta + \cos^2\theta)\,d\theta$$

$$= \theta\Big|_0^{\pi} - 2\sin\theta\Big|_0^{\pi} + \int_0^{\pi} \cos^2\theta\,d\theta$$

$$= \pi + \int_0^{\pi} \left(\frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= \pi + \int_0^{\pi} \tfrac{1}{2}\,d\theta + \int_0^{\pi} \frac{\cos 2\theta}{2}\,d\theta$$

$$= \pi + \frac{\theta}{2}\Big|_0^{\pi} + \frac{1}{4}\sin 2\theta\Big|_0^{\pi} = \frac{3}{2}\pi \qquad \left(\approx \frac{3}{2}\times 3 = 4.5\right)$$

#7 a) $\displaystyle\int \frac{\ln x}{x^2}\,dx$ $\quad\left(\begin{array}{ll} \text{let } u = \ln x & dv = x^{-2}dx \\ du = \frac{1}{x}dx & v = -x^{-1} \end{array}\right)$

$$= -\frac{\ln x}{x} + \int x^{-2}\,dx = -\frac{\ln x}{x} - \frac{1}{x} + C$$

b) $\displaystyle\int_1^{\infty} \frac{\ln x}{x^2}\,dx = \lim_{l\to\infty} \int_1^{l} \frac{\ln x}{x^2}\,dx$

$$= \lim_{l\to\infty}\left(-\frac{\ln x}{x} - \frac{1}{x}\right)\Big|_1^{l} = \lim_{l\to\infty}\left(\frac{-\ln l}{l} - \frac{1}{l}\right) + \left(\frac{\ln 1}{1} + \frac{1}{1}\right)$$

$\ln 1 = 0$

$\left(\displaystyle\lim_{l\to\infty} \frac{\ln l}{l} = \lim_{l\to\infty} \frac{\frac{1}{l}}{1} \text{ by l'Hopital's Rule} = 0\right)$

$$\therefore \int_0^{\infty} \frac{\ln x}{x}\,dx = 1$$

c) The integrand is non-negative everywhere on the domain of integration

#8 a) $\int \frac{du}{u^2-6u+5} = \int \frac{du}{(u-5)(u-1)}$ $\left[\frac{A}{u-5} + \frac{B}{u-1} = \frac{1}{(u-5)(u-1)} \quad A(u-1)+B(u-5)=1 \quad u=5 \Rightarrow A = 1/4,\ u=1 \Rightarrow B=-1/4 \right]$

$= \frac{1}{4}\left[\int \frac{du}{u-5} - \int \frac{du}{u-1}\right]$

$= \frac{1}{4}\ln|u-5| - \frac{1}{4}\ln|u-1| + C = \ln\left(\sqrt[4]{\left|\frac{u-5}{u-1}\right|}\right) + C$

b) $\int x e^{-2x}\,dx$ (Let $u=x$, $dv=e^{-2x}dx$. Then $du=dx$, $v=-\frac{1}{2}e^{-2x}$)

$= -\frac{1}{2}xe^{-2x} + \frac{1}{2}\int e^{-2x}dx = -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C$

$= -\frac{1}{4}(2x+1)e^{-2x} + C$

c) $\int \tan^6 z \sec^2 z\,dz$ (let $u=\tan z$, $du=\sec^2 z\,dz$)

$= \int u^6 du = \frac{1}{7}u^7 + C = \frac{1}{7}\tan^7 z + C$

#9 a) $\int \frac{dx}{\sqrt{x}(x+4)}$ (Let $u^2=x \Rightarrow 2u\,du=dx$)

$= \int \frac{2u\,du}{u(u^2+4)} = 2\int \frac{du}{u^2+4} = 2\int \frac{du}{4\left(\left(\frac{u}{2}\right)^2+1\right)} = \frac{1}{2}\int \frac{du}{(u/2)^2+1}$

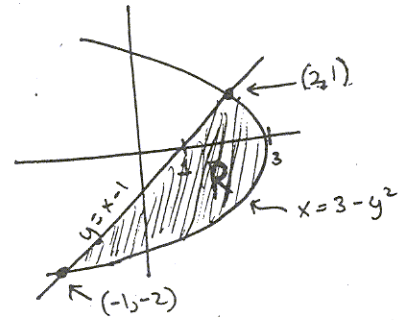
$= \int \frac{dv}{v^2+1}$ $\begin{pmatrix} v=u/2 \\ dv=du/2 \end{pmatrix}$ $= \tan^{-1}v + C = \tan^{-1}\frac{u}{2} + C$

$= \arctan\left(\frac{\sqrt{x}}{2}\right) + C$

b) $\int_{12}^{\infty} \frac{dx}{\sqrt{x}(x+4)} = \lim_{\ell\to\infty} \int_{12}^{\ell} \frac{dx}{\sqrt{x}(x+4)}$

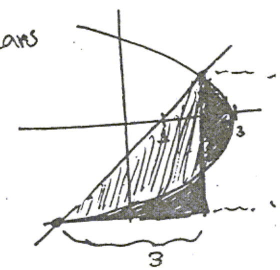
$= \lim_{\ell\to\infty} \tan^{-1}\frac{\sqrt{\ell}}{2} - \tan^{-1}\frac{\sqrt{12}}{2} = \frac{\pi}{2} - \tan^{-1}\sqrt{3} = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$

#10 a)



(Set the two equations equal to get points of intersection:
$1+y = 3-y^2 \Rightarrow y^2+y-2=0 \Rightarrow (y+2)(y-1)=0 \Rightarrow y=-2,1$
$y=-2 \Rightarrow x=-2+1=-1$, $y=1 \Rightarrow x=1+1=2$)

b) The area of the shaded region appears to be more or less the same as that of the right triangle with vertices $(-1,-2)$, $(1,2)$, $+(1,-2)$ — say $\frac{9}{2} \pm 1$



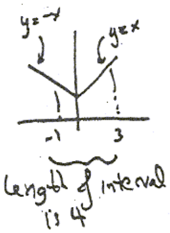
c) Integrating along y-axis take least effort, so

$\text{Area} = \int_{-2}^{1} \left[(3-y^2)-(y+1)\right]dy = \int_{-2}^{1}(2-y-y^2)\,dy$

$= \left(2y - \frac{y^2}{2} - \frac{y^3}{3}\right)\Big|_{-2}^{1} = \left(2-\frac{1}{2}-\frac{1}{3}\right) - \left(-4-2+\frac{8}{3}\right)$

$= 8 - \frac{1}{2} - \frac{9}{3} = \frac{9}{2}$

#11



$$f_{aver} = \frac{1}{4}\int_{-1}^{3} (2+|x|)\,dx$$

$$= \frac{1}{4}\int_{-1}^{3} 2\,dx + \frac{1}{4}\int_{-1}^{3} |x|\,dx$$

$$= \frac{1}{2}x\Big|_{-1}^{3} + \frac{1}{4}\left(\int_{-1}^{0} -x\,dx + \int_{0}^{3} x\,dx\right)$$

$$= \frac{1}{2}(3-(-1)) - \frac{1}{4}\frac{x^2}{2}\Big|_{-1}^{0} + \frac{1}{4}\frac{x^2}{2}\Big|_{0}^{3}$$

$$= \frac{1}{2}(4) - \frac{1}{8}(0-(-1)^2) + \frac{1}{8}(3^2-0^2) = 2 + \frac{1}{8} + \frac{9}{8} = \frac{13}{4}$$

(Alternative method: Since the region consists of two trapezoids, you can solve this problem with no calculus.

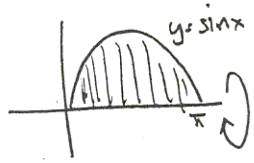
Area of left trapezoid $= \left(\frac{2+3}{2}\right)(1) = \frac{5}{2}$

Area of right trapezoid $= \left(\frac{2+5}{2}\right)(3) = \frac{21}{2}$

$\therefore$ total Area $= \frac{5}{2} + \frac{21}{2} = 13$

$\therefore$ Aver value $= \frac{\text{total Area}}{\text{total length of interval}} = \frac{13}{4}$)

#12



$$\text{Volume} = \int_0^{\pi} \pi \sin^2 x\,dx \quad \left(\begin{matrix}\text{disk}\\ \text{method}\end{matrix}\right)$$

$$= \int_0^{\pi} \pi\left(\frac{1-\cos 2x}{2}\right)dx = \frac{\pi}{2}\left(x - \frac{\sin 2x}{2}\right)\Big|_0^{\pi} = \frac{\pi^2}{2}$$

#13
a) T

b) F (consider $a_n = (-1)^n$)

c) F (consider $\sum \frac{(-1)^n x^n}{(2^n)\,n}$)

d) T (Radius of convergence is at least 2 $\therefore$ series converges for at least $|x|<2$)

e) T (for convergence, only the tail matters)

f) F (the inequality is in the wrong direction)

g) T ($\lim_{x\to 0^+} \frac{\cos x}{x} = \infty$)

h) F (eight petals)

i) T (see Problem #5)

j) T (the best 3rd deg. poly. approx. to a 3rd degree poly. is of course itself)