Calculus II - Final Examination - Fall 2005
December 13, 4-6 PM
no calculators
Please check your answers and explain your reasoning.
The questions with a star are worth 20 points, the others are 10 points.

1. Find the sum of the series
   \[ 2 - \frac{8}{3!} + \frac{32}{5!} - \frac{128}{7!} + \ldots. \]

2. \* Let \( f(x) = \frac{1}{3-x} \). Find the Taylor-MacLaurin series at \( x = 0 \) for \( f(x) \) and find
   the interval where the series converges.

3. Determine if the sequence
   \[ \left\{ \frac{n}{3n+1} \right\}_{n=0}^{\infty} \]
   is strictly increasing, strictly decreasing, both, or neither.
   If the sequence converges, find the limit.

4. \* Does the series
   \[ \sum_{k=1}^{\infty} \frac{k^2}{3^k} \]
   converge or diverge? Why? If it converges is the convergence absolute or conditional?

5. Convert the curve \( r = 2\sin\theta + 2\cos\theta \) into \( x,y \) coordinates and identify the type of curve. Then graph the curve.

6. \* Consider the area enclosed by the cardioid \( r = 1 - \cos\theta \). Without calculating
   which of the following answer(s) might be reasonable? Why? \(-498, 4, .0092, 6, 18, 498\). Finish by computing the actual area. If your answer does not seem
   reasonable please explain why you think this is.

7. \* Evaluate the following integrals:
   a. \[ \int \frac{\ln x}{x^2} \, dx \]
   b. \[ \int_{1}^{\infty} \frac{\ln x}{x^2} \, dx \]
   c. Why would a negative answer for 7.b make you suspicious?

8. \* Find the following integrals
a. \( \int \frac{du}{u^2 - 6u + 5} \) 

b. \( \int xe^{-2x} dx \) 

c. \( \int \tan^6 x \sec^2 x \, dz \)

9. * Now try

a. \( \int \frac{dx}{\sqrt{x(3+x)}} \) Hint: \( u^2 = x \)

b. \( \int_0^\infty \frac{dx}{\sqrt{x}(x+4)} \)

10. * Let \( R \) be the region enclosed by the equations \( y = x - 1 \) and \( x = 3 - y^2 \).

a. Sketch the region \( R \) and find the points of intersection of the two curves.

b. Just looking at your sketch give a range of values for the area.

c. Find the area enclosed by \( R \).

11. Find the average value of \( f(x) = 2 + |x| \) on the interval \([-1, 3]\).

12. * Find the volume generated when the area under one arch of the curve \( y = \sin x \) is revolved about the \( x \)-axis.

13. For each of the following statements, indicate whether it is True or False. For these problems a T or F suffices but a point will deducted for each wrong answer.

a. If \( \lim_{n \to \infty} a_n = L \) then \( \lim_{n \to \infty} a_{n+1} = L \)

b. If \( \lim_{n \to \infty} a_{2n} = L \) then \( \lim_{n \to \infty} a_n = L \)

c. If \( \sum_{n=0}^{\infty} a_n x^n \) converges at \( x = 2 \), then it also converges at \( x = -2 \)

d. If \( \sum_{n=0}^{\infty} a_n x^n \) converges at \( x = 2 \), then it also converges at \( x = -1 \)

e. If \( \sum_{n=10,000}^{\infty} a_n \) converges, then so does \( \sum_{n=1}^{\infty} a_n \)

f. If \( 0 \leq a_n \leq b_n \) for all \( n \), and if \( \sum_{n=1}^{\infty} b_n \) diverges, then \( \sum_{n=1}^{\infty} a_n \) also diverges.

g. \( \int_0^1 \frac{\cos x}{x} \) is an improper integral.

h. The graph of \( r = \cos 4\theta \) is a rose with 4 petals.

i. The graph of \( r = 2\cos \theta \) is a circle.

j. The Maclaurin polynomial of order 3 for \( f(x) = 2x^3 - x^2 + 7x - 11 \) is the same as \( f(x) \).
Alternative method: \[
\frac{a_n}{a_{n+1}} = \frac{3n+2}{3n+4} = \frac{3n+2}{3n+4} > 1
\]

For all \( n \) in an increasing sequence.

Consider \( f(x) = \frac{3x}{x+1} \).

This geometric series converges for \( \frac{3}{1} < 1 \) or \( 1 < 3 \).

\( r = \frac{3}{2} \) and \( r > 1 \) for case 2.

\[ r = 2 \sin \theta \]

\[ r^2 = 2r \sin \theta + r \cos \theta \]

Circle centered at \((0,1)\) with radius \(\sqrt{2} \).
The domain of integration is non-negative everywhere.

\[
\int_0^\infty \frac{e^{-\frac{x}{a^2}}}{y^2} \, dx = 1 \quad \text{(i)}
\]

(\text{L'Hopital's Rule})

\[
\lim_{a \to 0} \left( \frac{1}{1 + \frac{x}{a^2}} \right) = \lim_{a \to 0} \frac{\frac{1}{a^2}}{1 + \frac{x}{a^2}} = 0
\]

Therefore, the only reasonable statement on the given.

So, it appears that a rod of length 3 and height 2.5 with a rectangular cross-section.

Also, it appears that a rectangular approximated of dimension

inside of a rectangle of width 2.5 and height 3.

From the plot, it appears that the curve fits...
The given problem: Solve the region consisting of 

\[
\frac{1}{4} = \frac{\text{total length}}{\text{total area}}.
\]

\[
\frac{7}{13} + \frac{3}{5} = 13,
\]

Area of right trapezoid = \( \frac{7}{5} + \frac{3}{5} \).

Area of left triangle = \( \frac{7}{5} \).

An alternative method: Since the region consists of 

\[
\int_0^1 f(x) \, dx + \int_1^2 f(x) \, dx = \int_0^2 f(x) \, dx
\]

\[
\int_0^2 f(x) \, dx = \int_0^1 \left( x^2 + 2x - 1 \right) \, dx + \int_1^2 \left( -x^2 + 2x + 1 \right) \, dx
\]

\[
\int_0^1 (x^2 + 2x - 1) \, dx + \int_1^2 (-x^2 + 2x + 1) \, dx
\]

\[
\frac{1}{12}
\]