CALCULUS III (MATH-158)

FALL 2005 Final

Maximum 200 points

PART A. Answer any 5 problems in this part. Each problem carries 20 points.

I. Find parametric equations to the line of intersection of the planes
$2x + y - 3 = 0$ and $x + 2y + z = 0$
[*Hint: A direction vector of the line of intersection is perpendicular to the normals to the planes.*]

II. Sketch the region enclosed by the surfaces

$$z = x^2 + y^2$$
and
$$z = 4 - x^2 - y^2$$

What is the curve of intersection?

III. The equation to a surface in space in rectangular Cartesian coordinates is $z = 3x^2 + 3y^2$. Write the equation in cylindrical and then in spherical coordinates.

IV. Find the unit tangent, normal and binormal vectors for the curve

$$\mathbf{r(t)} = \langle e^t \sin t, e^t \cos t, 3 \rangle \quad \text{at } t=\pi.$$

V. Let $f(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$. Determine the limit of f(x, y) as $(x, y) \to (0,0)$ along the

[a] x axis
[b] y axis
[c] Does $\lim_{(x,y)\to(0,0)} f(x, y)$ exist? Justify your answer.
[d] Does the $\partial/\partial x\ f(x, y)$ exist at (0, 0)?

VI. [a] In what direction does $f(x, y) = x^2 + y^2$ increase the fastest at $(\sqrt{2}, \sqrt{2})$?
[b] Use your work in [a] to find the rate of change of $f(x, y) = x^2 + y^2$ at $(\sqrt{2}, \sqrt{2})$ in the direction <1, -1>.

VII. Let r, h denote radius and height of a right circular cylinder. Determine the dimensions that maximize the volume of a right circular cylinder with top and bottom having a surface area of 96π m^2.
[Hint: Recall Volume $=\pi r^2 h$ and Lateral Surface Area $=2\pi rh$.]

VIII. Find the volume inside the sphere $x^2 + y^2 + z^2 = 9$ and outside the cylinder $x^2 + y^2 = 1$.

IX. Evaluate the line integral $\int_C \frac{x}{1+y^2}$ ds with respect to arc length s along the parametric curve C : x = 1+ 2t, y= t $(0 \le t \le 1)$.

X. Change the integral $\int_0^2 \int_0^{\sqrt{4-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}} z^2 dzdxdy$ to equivalent integrals in cylindrical coordinates and then in spherical coordinates.(Do not evaluate the integrals.)

PART B. Answer any 4 problems in this part. Each problem carries 25 points.

XI. The equation to a conic in the x'y'-coordinate system is

$$x^2 - xy + y^2 - 2 = 0$$

Find a new x'y'-coordinate system in which the above equation is transformed to an equation without the x'y'-term. Then name the conic and sketch its graph.

XII. Find an equation to the plane determined by the points A(0, -2, 1), B(1, -1, -2) and C(-1, 1, 0).

XIII. Find the curvature and radius of curvature of the curve $x^2 + 4y^2 = 16$ at the point (4, 0), and draw the circle of curvature at this point.
[*Hint: Parameterize:* $x = 4 \cos t,\ y = 2 \sin t$.]

XIV. The position vector of a moving particle is

$$r(t) = \langle 2, t^3, -8 \ln t \rangle$$

Find the velocity, the speed, the acceleration, and the scalar tangential and normal components of the acceleration.

XV. [a] Find an equation of the tangent plane at the point (1, 0, 1) to the surface defined by $z = 1 - x^2 + \cos xy$.

[b] Find the parametric equations of the line perpendicular to the surface in [a] at the point (1, 0, 1).

XVI. Verify that (0, 0) and (1, 1) are critical points of $f(x, y) = x^3 - 3xy + y^3$ and use the second derivative partials test to classify the extrema as absolute/relative minima, maxima or saddle points.

XVII. Find the surface area of the portion of the sphere $x^2 + y^2 + z^2 = 16$ between the planes z = 1 and z = 2.

XVIII. Use Green's theorem to evaluate the integral $\oint_C \cos x \sin y dx + \sin x \cos y dy$ along the triangle C with vertices (0,0), (3,3) and (0,3). Assume C is positively oriented.