CALCULUS III (MATH-158)
FALL 2005 Final
Maximum 200 points

PART A. Answer any 5 problems in this part. Each problem carries 20 points.

I. Find parametric equations to the line of intersection of the planes
2x + y - 3 = 0 and x + 2y + z = 0
[Hint: A direction vector of the line of intersection is perpendicular to the normals to the planes.]

II. Sketch the region enclosed by the surfaces
z = x^2 + y^2
and
z = 4 - x^2 - y^2
What is the curve of intersection?

III. The equation to a surface in space in rectangular Cartesian coordinates is
z = 3x^2 + 3y^2. Write the equation in cylindrical and then in spherical coordinates.

IV. Find the unit tangent, normal and binormal vectors for the curve
\[ \mathbf{r}(t) = \langle e^t \sin t, e^t \cos t, 3 \rangle \text{ at } t=\pi. \]

V. Let \( f(x, y) = \frac{x^2 - y^2}{x^2 + y^2} \). Determine the limit of \( f(x, y) \) as \( (x, y) \to (0,0) \) along the
[a] x axis
[b] y axis
[c] Does \( \lim_{(x,y)\to(0,0)} f(x,y) \) exist? Justify your answer.
[d] Does the \( \frac{\partial}{\partial x} f(x,y) \) exist at \( (0, 0) \)?

VI. [a] In what direction does \( f(x, y) = x^2 + y^2 \) increase the fastest at \( (\sqrt{2}, \sqrt{2}) \)?
[b] Use your work in [a] to find the rate of change of \( f(x, y) = x^2 + y^2 \) at \( (\sqrt{2}, \sqrt{2}) \) in the direction \( <1, -1> \).

VII. Let \( r, h \) denote radius and height of a right circular cylinder. Determine the
dimensions that maximize the volume of a right circular cylinder with top and bottom
having a surface area of \( 96\pi \text{ m}^2 \).
[Hint: Recall Volume =\( \pi r^2 h \) and Lateral Surface Area =\( 2\pi rh \).

VIII. Find the volume inside the sphere \( x^2 + y^2 + z^2 = 9 \) and outside the
cylinder \( x^2 + y^2 = 1 \).
IX. Evaluate the line integral \( \int \frac{x}{1 + y^2} \, ds \) with respect to arc length \( s \) along the parametric curve \( C: x = 1 + 2t, y = t \, (0 \leq t \leq 1) \).

X. Change the integral \( \int \int \left( \frac{4 - y^2}{x^2 + y^2} \right) \, dz \, dx \, dy \) to equivalent integrals in cylindrical coordinates and then in spherical coordinates. (Do not evaluate the integrals.)

PART B. Answer any 4 problems in this part. Each problem carries 25 points.

XI. The equation to a conic in the \( x'y' \)-coordinate system is
\[ x'^2 - xy' + y'^2 - 2 = 0 \]
Find a new \( x'y' \)-coordinate system in which the above equation is transformed to an equation without the \( x'y' \)-term. Then name the conic and sketch its graph.

XII. Find an equation to the plane determined by the points \( A(0, -2, 1) \), \( B(1, -1, -2) \) and \( C(-1, 1, 0) \).

XIII. Find the curvature and radius of curvature of the curve \( x^2 + 4y^2 = 16 \) at the point \((4, 0)\), and draw the circle of curvature at this point.
[Hint: Parameterize: \( x = 4 \cos t, y = 2 \sin t \).]

XIV. The position vector of a moving particle is
\( r(t) = <2, t^3, -8 \ln t> \)
Find the velocity, the speed, the acceleration, and the scalar tangential and normal components of the acceleration.

XV. [a] Find an equation of the tangent plane at the point \((1, 0, 1)\) to the surface defined by \( z = 1 - x^2 + \cos xy \).
[b] Find the parametric equations of the line perpendicular to the surface in [a] at the point \((1, 0, 1)\).

XVI. Verify that \((0, 0)\) and \((1, 1)\) are critical points of \( f(x, y) = x^3 - 3xy + y^3 \) and use the second derivative partials test to classify the extrema as absolute/relative minima, maxima or saddle points.

XVII. Find the surface area of the portion of the sphere \( x^2 + y^2 + z^2 = 16 \) between the planes \( z = 1 \) and \( z = 2 \).

XVIII. Use Green’s theorem to evaluate the integral \( \int \cos x \sin y \, dx + x \cos y \, dy \)
along the triangle \( C \) with vertices \((0,0)\), \((3,3)\) and \((0,3)\). Assume \( C \) is positively oriented.