CALCULUS III [MATH 158]
FALL 2006 FINAL

INSTRUCTIONS: Answer any 5 questions of PART A, and any 4 questions of PART B.

PART A.

Each problem in part A is worth 20 points. Answer any 5 of the following 10 questions.

1. Find an equation of the plane which passes through $(1, -2, -3)$ and is perpendicular to $x = 1 + 3t,\ y = 2 + t,\ z = -3 - 6t$.

2. **(i)** Sketch the region enclosed by the surfaces $z = x^2 + y^2$ and $z = 4 - x^2 - y^2$, and
(ii) describe their curve of intersection.

3. Recall that $(r, \theta, z), (\rho, \theta, \phi)$ denote the cylindrical coordinate and the spherical coordinates respectively. Describe and sketch the sets given below in (i), (ii) and (iii) in 3-space.
(i) $z = r\cos\theta$
(ii) $\rho\sin\phi = 2\cos\theta$
(iii) $r^2 \le z \le 4$.

4. Find the rotation angle needed to remove the xy-term in the equation $x^2 - xy + y^2 - 6 = 0$; then name the conic and give its equation in $x'y'$-coordinates after the xy-term is removed.

5. Find an arc length parameterization of the curve $\vec{r}_1(t) = \langle \cos^3 t,\ \sin^3 t\rangle$ when $0 \le t \le \pi/2$ that has the same orientation as the given curve and has $t = 0$ as the reference point.

6. Find the curvature $\kappa(t)$ and the radius of curvature at $t = \pi/2$ for the curve $\vec{r}(t) = (e\cos t)\vec{i} + (4\sin t)\vec{j} + t\vec{k}$.

7. Let $f(x, y) = \frac{2x^2 - y^2}{x^2 + 2y^2}$.
(i) Show that $f(x, y)$ approaches $\frac{1}{3}$ as $(x, y) \to (0, 0)$ along the straight line $y = x$.
(ii) Show that $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist by letting $(x, y) \to (0, 0)$ along the curve $y = x^2$.

8. The electrical potential V at (x, y, z) is given by $V = x^2 + 4y^2 + 9z^2$.
(i) Find the rate of change of V at $P(2, -1, 3)$ in the direction from P to the origin.
(ii) Find the direction that produces the maximum rate of change of V at P.
(iii) What is the maximum rate of change of V at P?

9. Find equations of the tangent plane and normal line to graph of $7z = 4x^2 - 2y^2$ at $P(-2, -1, 2)$.

10. (i) Reverse the order of integration and evaluate $\int_0^1 \int_y^{\sqrt{y}} 2xy \, dx \, dy$.
(ii) Sketch the region R of integration and label the curves.

PART B. Answer any 4 of the following 8 questions in this part. Each problem in PART B is worth 25 points.

11. (i) Show that the graphs of $\vec{r}_1(t) = (2e^{-t})\vec{i} + (\cos t)\vec{j} + (t^2 + 3)\vec{k}$ and $\vec{r}_2(t) = (1 - t)\vec{i} + (t^2)\vec{j} + (t^3 + 4)\vec{k}$ intersect at the point $P(2, 1, 3)$.
(ii) Find the acute angle between the tangent lines to the graphs of $\vec{r}_1(t)$ and $\vec{r}_2(t)$ at the point P.

12. Let $\vec{r}(t) = \langle a\cos t,\ b\sin t,\ ct\rangle$ $(a \neq 0,\ b \neq 0,\ c \neq 0)$. Use the formula $\vec{B}(t) = \vec{T}(t) \text{ x } \vec{N}(t)$ to find $\vec{B}(t)$.

13. Locate all relative extrema and saddle points of
$f(x, y) = x^3 + y^3 - 3xy$.

14. If $f(x, y, z) = 4x^2 + y^2 + 5z^2$, use the method of Lagrange multipliers to find the minimum of $f(x, y, z)$ on the plane $2x + 3y + 4z = 12$.

15. Use polar coordinates to find the surface area of the portion of the paraboloid $z = x^2 + y^2$ below the plane $z = 4$ and above the xy plane.

16. Set up (BUT DO NOT EVALUATE) the triple integral to find the volume of the sphere $x^2 + y^2 + z^2 = 9$ in the first octant TWO WAYS using:
(i) rectangular coordinates
and
(ii) spherical coordinates.

17. Evaluate the line integral

$$\int_C x^2 dx + xy dy + dz$$

where C is the curve defined by $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle = \langle 3t, t^2, 1 \rangle$.

18. Use Green's Theorem to evaluate the line integral

$$\oint_C (y^4 + x^3)dx + 2x^6 dy$$

where C is the boundary of the closed unit square formed by the vertices $\{(0,0),\ (0,1),\ (1,1)\ (0,1)\}$ (oriented counterclockwise).