Final Exam

Math 158 Calculus III

Fall, 2014

Please show all work.

Do 8 problems from part A and 4 problems from part B.

Part A

Please do 8 problems from part A. Each problem is worth 15 points

1. Let $f(x, y) = 4x^2 + 9y^2 + 1$.
   (a) Find the gradient $\vec{\nabla} f(1, -1)$.
   (b) Write an equation of a tangent line to the graph of $f(x, y) = 14$ at the point $(1, -1)$.
   (c) Write an equation of a normal line to the graph of $f(x, y) = 14$ at the point $(1, -1)$.

2. Write an equation for the plane containing the points $(1, 2, -3)$, $(2, -3, 4)$, and $(3, 4, 7)$.

3. A point moves on a trajectory $\vec{r}(t)$ so that $\vec{r}\,'(t) = (3\cos t)\,\vec{i} + (4\cos t)\,\vec{j} - (5\sin t)\,\vec{k}$ and $\vec{r}(0) = \vec{i} - \vec{j} + \vec{k}$.
   (a) Find $\vec{r}(t)$.
   (b) Find the speed at time t.
   (c) Find the acceleration at time t.

4. The function $z = f(x, y)$ is implicitly defined by the equation

$$x \sin z + y(1 + z) - \cos x - 2 = 0\,.$$

Evaluate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at the point $(\pi, 1, 0)$.

5. Let $f(x, y, z) = z^2 - e^{x^2+y}$ and let $\vec{u} = (-1, 2, 0)$. Find the directional derivative of f at $(0, 0, 1)$ in the direction of $\vec{u}$.

6. Let $f(x, y, z) = \ln(x + \sin z) + zy^2$. Find an equation of the plane tangent to the surface $f(x, y, z) = 0$ at the point $(1, 2, 0)$.

7. Find the points on the sphere $x^2 + y^2 + z^2 = 11$ that are closest to and farthest from the point $(3, 1, -1)$. (The easiest way to do this is probably to use Lagrange multipliers.)

8. Evaluate the integral $\displaystyle\int_0^3 \int_{y^2}^9 y e^{-x^2}\, dx\, dy$.

Please turn over.

9. Let R be the portion of the disk $\{(x,y) : x^2+y^2 \le 4\}$ which lies in the second quadrant. Evaluate $\iint_R \sqrt{4-x^2-y^2}\, dA$. (You will probably find it useful to use polar coordinates.)

10. Evaluate the integral $\int_0^1 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_0^{2x+y} 1\, dz\, dx\, dy$.

11. Let $\vec{r}(t) = (4\cos t)\,\vec{\imath} + (4\sin t)\,\vec{\jmath} + (3t)\,\vec{k}$ and let $C = \{\vec{r}(t) : -2\pi \le t \le 2\pi\}$. Evaluate the line integral $\int_C \sqrt{x^2+y^2}\, ds$.

12. Let $\vec{F}(x,y,z) = (1+y^2-e^z, \sin z + 2xy, y\cos z - xe^z)$. Find a function f such that $\vec{F} = \vec{\nabla} f$.

Part B

Please do 4 problems from part B. Each problem is worth 20 points

13. Find the arc length of the parametric curve C given by: $x=t^3$, $y=t$, $z=\frac{\sqrt{6}}{2}t^2$, $1 \le t \le 3$.

14. Let $f(x,y) = \dfrac{x^2y}{x^4+2y^2}$.
   (a) Show that $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along any straight line.
   (b) Find the limit of $f(x,y)$ as $(x,y) \to (0,0)$ along the parabola $y=x^2$.
   (c) Does $\lim_{(x,y)\to(0,0)} f(x,y)$ exist? Justify your answer.

15. Use the $(\epsilon - \delta)$ definition of the limit to prove that $\lim_{(x,y)\to(0,0)} x^2+4y^2 = 0$.

16. Find all local maxima, minima, and saddle points of the function

$$f(x,y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 3.$$

17. Let R be the region in the plane bounded by the lines $y = x-1$, $y = 1-x$, and $x=3$. Find the centroid of R. (You should be able to figure out the area of the region and $\bar{y}$ without calculating any integrals.)

18. Let $\vec{F}(x,y) = (e^{2x}+2y, xy+\sin^3 y)$ and let C be the triangle with vertices $(0,0)$, $(1,0)$, and $(1,1)$. Find the work done if the force $\vec{F}$ is applied to a particle traversing C in the counterclockwise direction. That is, evaluate

$$\oint_C \vec{F}(\vec{r}) \cdot d\vec{r} = \oint_C (e^{2x}+2y)\,dx + (xy+\sin^3 y)\,dy.$$

(This is an easy problem if you use Green's Theorem.)