Final Exam
Math 159 Calculus III
Spring 2006

Choose only 8 questions from #1 - #12 and only 7 questions from #13 - #21.

[20 pts] 1. Find the maximum and minimum values of the radius of curvature $\rho$ for the
curve $x = \cos t$, $y = \sin t$, $z = \sin t$; $0 \leq t < 2\pi$.
(Note: $\rho(t) = 1/\kappa(t)$ where $\kappa$ denotes the curvature.)

[20 pts] 2. Evaluate the double integral
$$\iint_R \sqrt{x^2 + y^2} \, dA,$$
where $R = \{(x, y) : x^2 + y^2 \leq 1\}$.

[20 pts] 3. Evaluate the double integral
$$\iint_R \frac{xy}{\sqrt{1 + x^2 + y^2}} \, dA,$$
where $R = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$.

[20 pts] 4. Find the surface area of the surface $z = 1 - x^2 - y^2$ with $1 - x^2 - y^2 \geq 0$.

[20 pts] 5. Evaluate the iterated integral
$$\int_0^{2\pi} \int_0^1 \int_0^{\sqrt{1-r^2}} zr \, dz \, dr \, d\theta.$$

[20 pts] 6. Use the transformation $u = x - 2y$, $v = 2x + y$ to find
$$\iint_R \frac{x - 2y}{2x + y} \, dA,$$
where $R$ is the rectangular region enclosed by the lines
$x - 2y = 1$, $x - 2y = 4$, $2x + y = 1$, $2x + y = 3$.

[20 pts] 7. Use Lagrange multipliers to find the maximum and the minimum values of
$f(x, y, z) = xyz$ subject to the condition $x^2 + y^2 + z^2 = 1$.

[20 pts] 8. Find the local maxima, minima, and saddle points of $f(x, y) = e^{-(x^2+y^2+2x)}$.

[20 pts] 9. Find the equation of the tangent plane of $z = \ln(\sqrt{x^2 + y^2})$ at $(-1, 0, 0)$.

[20 pts] 10. Find the unit tangent and unit normal vectors to the graph of the curve
$r(t) = \ln t \, \hat{i} + t \, \hat{j}$ at $P(0, 1)$. Sketch the curve showing the point of tangency.
(Be careful in drawing the direction of the unit tangent and the unit normal vectors.)

[20 pts] 11. Find the unit vector in the direction in which $f(x, y, z) = \tan^{-1}\left(\frac{x}{y + z}\right)$
increases most rapidly at $(4, 2, 2)$.
(Note: $\frac{d}{du}(\tan^{-1}u) = \frac{1}{1+u^2}$)
12. (a) Express the vector \( \mathbf{v} = \langle -1, 4, 8 \rangle \) as the sum of two orthogonal vectors such that one of them is parallel to \( \mathbf{b} = \langle 2, -2, -1 \rangle \). Use your decomposition to compute the distance from the point \((-1, 4, 8)\) to the line determined by the vector \( \mathbf{b} \).

(b) Show that in 3-space the distance \( d \) from a point \( P \) to the line \( L \) that is passing through the points \( A \) and \( B \) can be given by the formula

\[
d = \frac{\| \overrightarrow{AP} \times \overrightarrow{AB} \|}{\| \overrightarrow{AB} \|}
\]

13. Find the directional derivative of \( f(s, y, z) = \sin(xyz) \) at \((1/2, 1/2, \pi)\) in the direction of \( \langle 1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3} \rangle \).

14. Find \( f_x \) for \( f(x, y) = \int_1^{xy} e^t \, dt \).

15. (a) Find parametric equations of the directed line segment from the point \( P(2,1,3) \) to the point \( Q(1,3,-2) \).

(b) Use vectors to determine whether the points \( P_1(3,1,3) \), \( P_2(1,5, -1) \) and \( P_3(4,-1,5) \) are collinear.

16. Let \( \mathbf{r} = \langle x, y \rangle \), and fix two distinct points \( \mathbf{r}_1 = \langle x_1, y_1 \rangle \) and \( \mathbf{r}_2 = \langle x_2, y_2 \rangle \). Given \( a > 0 \) and \( \| \mathbf{r}_2 - \mathbf{r}_1 \| > a \), describe the set of all points \((x, y)\) for which \( \| \mathbf{r} - \mathbf{r}_2 \| - \| \mathbf{r} - \mathbf{r}_1 \| = a \). (Do not attempt to derive the equation in standard form algebraically.)

17. (a) Let \( A \), \( B \), and \( C \) be three distinct noncollinear points in 3-space. Describe the set of points \( P \) that satisfy the vector equation \( \overrightarrow{AP} \cdot (\overrightarrow{AB} \times \overrightarrow{AC}) = 0 \).

(b) Determine whether the points \( A(0,0,0) \), \( B(1, -1, 1) \), \( C(2,1, -2) \) and \( D(-1, 2, -1) \) are coplanar, (i.e., lie on the same plane).

18. Determine whether the line \( L_1: \ x = 3 - t, \ y = 5 + 3t, \ z = -1 - 4t \) and the line \( L_2: \ x = 8 + 2t, \ y = -6 - 4t, \ z = 5 + 2t \) have a point of intersection.

19. Find the parametric equations of the line through \((2, 0, -3)\) that is parallel to the line of intersection of the planes \( x + 2y + 3z + 4 = 0 \) and \( x - y - z - 5 = 0 \).

20. Find parametric equations for \( \mathbf{r} = \langle 2 + \cos 3t, 3 - \sin 3t, 4t \rangle, \ 0 \leq t \leq \pi/4 \), using arc length \( s \) as a parameter. Take the point on the curve where \( t = 0 \) as the reference point.

21. Find the work done by a force field \( \mathbf{F}(x, y) = \langle x^3 - y^3, xy^2 \rangle \) along the curve \( C \) parametrized by \( x = t^2, \ y = t^3, \ -1 \leq t \leq 0 \).