Calc 3 Final Exam (Spring 06)
Solutions

#1 $r(t) = \langle \cos t, \sin t, \sin t \rangle, \quad 0 \le t \le 2\pi$

$$K(t) = \frac{\|r'(t) \times r''(t)\|}{\|r'\|^3}$$

$r'(t) = \langle -\sin t, \cos t, \cos t \rangle$

$r''(t) = \langle -\cos t, -\sin t, -\sin t \rangle$

$r'(t) \times r''(t) = \langle 0, -1, 1 \rangle \Rightarrow \|r' \times r''\| = \sqrt{2}$

$\|r'(t)\|^3 = (\sin^2 t + 2\cos^2 t)^{3/2} = (1 + \cos^2 t)^{3/2}$

$$\therefore \rho(t) = \frac{1}{K(t)} = \frac{(1+\cos^2 t)^{3/2}}{\sqrt{2}}$$

By inspection it is clear that the max occurs when $\cos t = 1$ and the min occurs when $\cos t = 0$, but we can proceed mechanically through our Calculus I routine...

$\rho'(t) = \left(\frac{3}{2\sqrt{2}}\right)(1 + \cos^2 t)^{1/2} \cdot 2\cos t \sin t = 0$

$\Rightarrow$ critical pts on the interval $[0, 2\pi)$ are $t = 0, \frac{\pi}{2}, \pi, \frac{3}{2}\pi$

$\rho(0) = \rho(\pi) = 2^{3/2}/2^{1/2} = 2$ max

$\rho(\frac{\pi}{2}) = \rho(\frac{3}{2}\pi) = 1/\sqrt{2}$ min

#2 $\displaystyle\iint_R \sqrt{x^2+y^2}\, dA \qquad R = \{(x,y) : x^2+y^2 \le 1\}$

$$= \int_0^{2\pi}\int_0^1 r \; r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^1 r^2\,dr = \frac{2\pi}{3}$$

#3 $\displaystyle\iint_R \frac{xy}{\sqrt{1+x^2+y^2}}\, dA \qquad R = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le 1\}$

$$\int_0^1 y \left(\int_0^1 \frac{x}{\sqrt{1+x^2+y^2}}\, dx \right) dy$$

(Inner integral: let $u = 1 + x^2 + y^2$, $du = 2x\,dx$

$\displaystyle\int \frac{x}{\sqrt{1+x^2+y^2}}\, dx = \int u^{-1/2} \frac{du}{2} = u^{1/2}$)

$$= \int_0^1 y \left(\sqrt{1+x^2+y^2}\,\Big|_{x=0}^{x=1} \right) dy = \int_0^1 y\left(\sqrt{2+y^2} - \sqrt{1+y^2}\right) dy$$

$$= \int_0^1 \sqrt{2+y^2}\; y\,dy - \int_0^1 \sqrt{1+y^2}\; y\,dy$$

let $u = 2+y^2$, etc.   let $u = 1+y^2$, etc.

$$= \frac{(2+y^2)^{3/2}}{3}\Big|_0^1 - \frac{(1+y^2)^{3/2}}{3}\Big|_0^1 = \left(\frac{3^{3/2}}{3} - \frac{2^{3/2}}{3}\right) - \left(\frac{2^{3/2}}{3} - \frac{1}{3}\right)$$

$$= \frac{1}{3}\left(3^{3/2} - 2^{5/2} + 1\right)$$

#4 $S = \iint_R \sqrt{1+f_x^2+f_y^2}\, dA$ where $f(x,y) = 1-x^2-y^2$ and $R = \{(x,y) : x^2+y^2 \le 1\}$

$$= \iint_R \sqrt{1+(-2x)^2+(-2y)^2}\, dA = \iint_R \sqrt{1+4(x^2+y^2)}\, dx\,dy$$

$$= \int_0^{2\pi}\int_0^1 \sqrt{1+4r^2}\, r\,dr\,d\theta = 2\pi \int_0^1 (1+4r^2)^{1/2}\, r\,dr$$

$\left(\begin{array}{l}\text{Let } u = 1+4r^2 \\ du = 8r\,dr\end{array}\right)$

$$= 2\pi \left(\frac{2}{3}\right)\left(\frac{1}{8}\right)(1+4r^2)^{3/2}\Big|_0^1 = \frac{\pi}{6}\left(5^{3/2}-1\right)$$

#5 $\int_0^{2\pi}\int_0^1\int_0^{\sqrt{1-r^2}} z r\, dz\,dr\,d\theta$

$$= \int_0^{2\pi}\int_0^1 r\, \frac{z^2}{2}\Big|_0^{\sqrt{1-r^2}} dr\,d\theta = \int_0^{2\pi}\int_0^1 \frac{r(1-r^2)}{2}\, dr\,d\theta$$

$$= 2\pi \cdot \frac{1}{2}\left(\frac{r^2}{2} - \frac{r^4}{4}\right)\Big|_0^1 = \frac{\pi}{4}$$

#6 $\iint_R \frac{x-2y}{2x+y}\, dx\,dy$ R bounded by

$x-2y=1 \leftrightarrow u=1$

$x-2y=4 \leftrightarrow u=4$

$2x+y=1 \leftrightarrow v=1$

$2x+y=3 \leftrightarrow v=3$

$$= \int_1^3\int_1^4 \frac{u}{v}\left|J(u,v)\right| du\,dv, \text{ where } u = x-2y,\ v = 2x+y$$

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \det\begin{vmatrix} \frac{\partial x}{\partial u}, & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u}, & \frac{\partial y}{\partial v}\end{vmatrix} = 1 \Big/ \det\begin{vmatrix} \frac{\partial u}{\partial x}, & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x}, & \frac{\partial v}{\partial y}\end{vmatrix}$$

$$= 1 \Big/ \det\begin{vmatrix} 1, & -2 \\ 2, & 1\end{vmatrix} = \frac{1}{5}$$

$$= \int_1^3\int_1^4 \frac{u}{v}\left(\frac{1}{5}\right) du\,dv = \frac{1}{5}\left(\int_1^3 \frac{1}{v}\, dv\right)\left(\int_1^4 u\, du\right)$$

$$= \frac{1}{5} \ln v\Big|_1^3 \frac{u^2}{2}\Big|_1^4 = \frac{1}{5}(\ln 3)\left(\frac{15}{2}\right) = \frac{3}{2}\ln 3$$

#7

$f(x,y,z) = xyz$

$g(x,y,z) = x^2+y^2+z^2-1=0$

$\vec{\nabla} f(x,y,z) = \lambda \vec{\nabla} g(x,y,z) \Rightarrow$

$$\left\{\begin{array}{l} yz=\lambda 2x \\ xz=\lambda 2y \\ xy=\lambda 2z \end{array}\right\} \Rightarrow \frac{yz}{x}=\frac{xz}{y}=\frac{xy}{z} \Rightarrow \begin{array}{l} x=\pm y \\ y=\pm z \end{array}$$

$\therefore (\pm x)^2+(\pm x)^2+(\pm x)^2-1=0 \Rightarrow 3x^2=1 \Rightarrow x=\pm\frac{1}{\sqrt{3}}$

$\therefore$ critical pts are $(x,y,z) = \left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right)$

$\Rightarrow$ max is $\frac{1}{3\sqrt{3}}$ min is $\frac{-1}{3\sqrt{3}}$

#8

$f(x,y) = e^{-x^2-y^2-2x}$

$\vec{\nabla} f = e^{-x^2-y^2-2x}\langle -2x-2, -2y\rangle$

$\Rightarrow \vec{\nabla} f = 0$ when $(x,y) = (-1,0)$

To save writing I will label $A := -x^2-y^2-2x$

$f_{xx} = e^A\left[(-2x-2)^2-2\right] = e^A\left(4(x+1)^2-2\right)$

$f_{yy} = e^A\left[(-2y)^2-2\right] = e^A\left[4y^2-2\right]$

$f_{xy} = e^A(-2x-2)(-2y) = e^A 4y(x+1)$

At $(x,y)=(-1,0)$: $f_{xx} = -2e$

$f_{yy} = -2e$

$f_{xy} = 0$

So $D = f_{xx}f_{yy} - f_{xy}^2 = 4e^2 > 0$ and $f_{xx} < 0$ $\Bigg\} \Rightarrow f(-1,0)$ is a local min

#9 $z = \ln\left(\sqrt{x^2+y^2}\right)$, $P(-1,0,0)$

$= \frac{1}{2}\ln(x^2+y^2)$

Normal vector to surface is $\left\langle \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1 \right\rangle$

$= \left\langle \frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}, -1 \right\rangle$

So at $P(-1,0,0)$, normal $\vec{n} = \langle -1, 0, -1 \rangle$

$\therefore$ equation of plane is $\vec{n} \cdot ((x,y,z) - P) = 0$

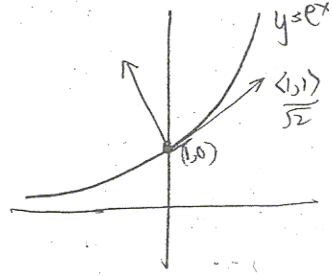
$\langle -1,0,-1 \rangle \cdot \langle x+1, y, z \rangle = 0$

$-(x+1) - z = 0 \Rightarrow x + z = -1$

#10 $\vec{r}(t) = \langle \ln t, t \rangle \Rightarrow \{x = \ln t, y = t\}$

At $P(0,1)$, $t=1$ $\Rightarrow x = \ln y \Rightarrow y = e^x$

$\vec{r}'(t) = \langle \frac{1}{t}, 1 \rangle \Rightarrow \vec{r}'(1) = \langle 1,1 \rangle \Rightarrow \vec{T}(1) = \frac{\langle 1,1 \rangle}{\sqrt{2}}$



$\therefore \vec{N}(1)$ must be either $\langle -1,1 \rangle/\sqrt{2}$ or $\langle 1,-1 \rangle/\sqrt{2}$

Since $\vec{N}$ points inwards,

$\vec{N}(1) = \langle -1,1 \rangle/\sqrt{2}$

Alternatively, we can compute $\vec{N}(1)$ methodically by the formula $N(1) = \vec{T}'(1)/\|\vec{T}'(1)\|$

$\vec{T}(t) = \frac{\langle t^{-1}, 1 \rangle}{(t^{-2}+1)^{1/2}} = \frac{\langle t^{-1}, 1 \rangle}{t^{-1}(1+t^2)^{1/2}} = \frac{\langle 1, t \rangle}{(1+t^2)^{1/2}}$

$\vec{T}'(t) = \frac{\langle 0,1 \rangle (1+t^2)^{1/2} - \langle 1,t \rangle \frac{1}{2}(1+t^2)^{-1/2}(2t)}{1+t^2}$

$\vec{T}'(1) = \frac{\langle 0,1 \rangle \sqrt{2} - \langle 1,1 \rangle (1/\sqrt{2})}{2} = \frac{2\langle 0,1 \rangle - \langle 1,1 \rangle}{2\sqrt{2}}$

$= \frac{\langle 0,2 \rangle - \langle 1,1 \rangle}{2\sqrt{2}} = \frac{\langle -1,1 \rangle}{2\sqrt{2}}$

$\therefore \frac{\vec{T}'(1)}{\|\vec{T}'(1)\|} = \frac{\langle -1,1 \rangle}{\sqrt{2}} = \vec{N}(1)$

#11 $f(x,y,z) = \tan^{-1}\left(\frac{x}{y+z}\right)$

$$\vec{\nabla} f(x,y,z) = \frac{(y+z)^2}{(y+z)^2 + x^2}\left\langle \frac{1}{y+z}, \frac{-x}{(y+z)^2}, \frac{-x}{(y+z)^2}\right\rangle$$

$$\vec{\nabla} f(4,2,2) = \frac{16}{2(16)}\left\langle \frac{1}{4}, \frac{-4}{4^2}, \frac{-4}{4^2}\right\rangle$$

$$= \frac{1}{2}\left\langle \frac{1}{4}, \frac{-1}{4}, \frac{-1}{4}\right\rangle = \frac{1}{8}\langle 1,-1,-1\rangle$$

$\therefore$ the unit vector is $\langle 1,-1,-1\rangle/\sqrt{3}$

#12 a) $\vec{u} = \langle -1,4,8\rangle$, $\vec{b} = \langle 2,-2,-1\rangle$,

$$\frac{\vec{b}}{\|\vec{b}\|} = \frac{\langle 2,-2,-1\rangle}{3} \Rightarrow \text{component of } \vec{u} \text{ in } \vec{b} \text{ direction is}$$

$$\left(\vec{u}\cdot\frac{\vec{b}}{\|\vec{b}\|}\right)\left(\frac{\vec{b}}{\|\vec{b}\|}\right) =$$

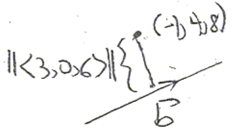
$$\left(\frac{\langle -1,4,8\rangle\cdot\langle 2,-2,-1\rangle}{9}\right)\langle 2,-2,-1\rangle$$

$$= \frac{-2-8-8}{9}\langle 2,-2,-1\rangle$$

$$= -2\langle 2,-2,-1\rangle = \langle -4,4,2\rangle$$

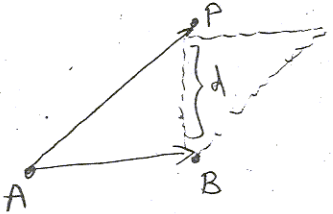
and orthogonal vector is $\langle -1,4,8\rangle - \langle -4,4,2\rangle = \langle 3,0,6\rangle$

Hence the distance of $(-1,4,8)$ to the line with direction vector $\vec{b}$ through the origin is



$$\|\langle 3,0,6\rangle\| = \sqrt{9+36} = \sqrt{45}$$

#12b



We know that the area of the parallelogram is equal to the base times the altitude

— i.e. $\|AB\| \cdot d$

But the area is also equal to $\|\vec{AP} \times \vec{AB}\|$

So $\|AB\| d = \|\vec{AP} \times \vec{AB}\|$

$$\Rightarrow d = \frac{\|\vec{AP} \times \vec{AB}\|}{\|AB\|}$$

#13 $f(x,y,z) = \sin(xyz)$

$$\vec{\nabla} f(x,y,z) = \cos(xyz) \langle yz, xz, xy \rangle$$

$$\vec{\nabla} f\left(\tfrac{1}{2}, \tfrac{1}{2}, \pi\right) = \cos\left(\tfrac{\pi}{4}\right) \left\langle \tfrac{\pi}{2}, \tfrac{\pi}{2}, \tfrac{1}{4} \right\rangle$$

$$= \frac{\sqrt{2}}{2} \left\langle \tfrac{\pi}{2}, \tfrac{\pi}{2}, \tfrac{1}{4} \right\rangle$$

$$= \frac{\sqrt{2}}{8} \langle 2\pi, 2\pi, 1 \rangle$$

If $\vec{u} = \frac{1}{\sqrt{3}} \langle 1,1,1 \rangle$

then $D_u f\left(\tfrac{1}{2}, \tfrac{1}{2}, \pi\right) = \vec{u} \cdot \vec{\nabla} f\left(\tfrac{1}{2}, \tfrac{1}{2}, \pi\right)$

$$= \left(\sqrt{\tfrac{2}{3}}\right)\left(\tfrac{1}{8}\right) \langle 1,1,1 \rangle \cdot \langle 2\pi, 2\pi, 1 \rangle$$

$$= \left(\frac{1}{4\sqrt{6}}\right)(4\pi + 1)$$

#14 $f(x,y) = \int_1^{xy} e^{t^2} dt$

$$\Rightarrow f_x(x,y) = e^{(xy)^2} \frac{\partial}{\partial x}(xy) = y e^{x^2 y^2}$$

#15 a) $\langle 2,1,3\rangle t + \langle 1,3,-2\rangle(1-t)$, $0 \le t \le 1$

$$\Rightarrow \left\{\begin{array}{l} x = 2t + (1-t) = t+1 \\ y = t + 3(1-t) = -2t+3 \\ z = 3t - 2(1-t) = 5t - 2 \end{array}\right\} 0 \le t \le 1$$

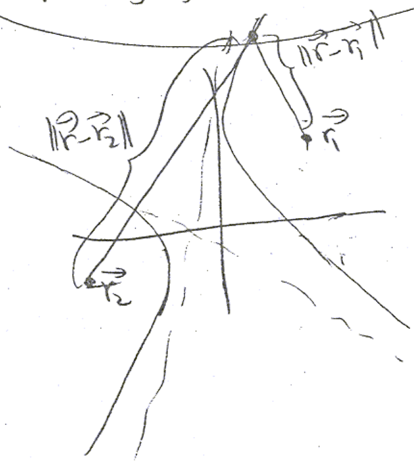
b) $\overrightarrow{P_1P_2} = (1,5,-1) - (3,1,3) = \langle -2,4,-4\rangle$

$\overrightarrow{P_2P_3} = (4,-1,5) - (1,5,-1) = \langle 3,-6,6\rangle$

$\left(-\frac{3}{2}\right)\overrightarrow{P_1P_2} = \left(-\frac{3}{2}\right)\langle -2,4,-4\rangle = \langle 3,-6,6\rangle = \overrightarrow{P_2P_3}$

$\therefore$ $P_1, P_2, + P_3$ are collinear.

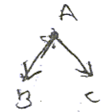
#16



$\|\vec{r}-\vec{r}_2\| - \|\vec{r}-\vec{r}_1\| = \text{const}$

$\Rightarrow$ locus of pts $\vec{r}$ forms a hyperbola

#17 a)



$\{P : \overrightarrow{AP}\cdot(\overrightarrow{AB}\times\overrightarrow{AC}) = 0\}$

is the plane containing the point A w/ normal $\vec{n} = \overrightarrow{AB}\times\overrightarrow{AC}$

b) $\overrightarrow{AD} = \langle -1,2,-1\rangle$

$\overrightarrow{AB} = \langle 1,-1,1\rangle$

$\overrightarrow{AC} = \langle 2,1,-2\rangle$

$\overrightarrow{AB}\times\overrightarrow{AC} = \langle 1,4,3\rangle$

$\therefore$ $\overrightarrow{AD}\cdot(\overrightarrow{AB}\times\overrightarrow{AC}) = \langle -1,2,-1\rangle\cdot\langle 1,4,3\rangle = -1+8-3 \ne 0$

$\therefore$ A, B, C + D are not coplanar.

#18 Is there a t + s such that $\langle 2,-4,2\rangle s + \langle 8,-6,5\rangle$

$\langle -1,3,-4\rangle t + \langle 3,5,-1\rangle = \langle 2,-4,2\rangle s + \langle 8,-6,5\rangle$

x components: $-t+3 = 2s+8$

y components: $3t+5 = -4s-6$

$\Rightarrow t+11 = 16$

$\Rightarrow t = 5$

$\therefore$ z components: $-4(5)-1 = -21$ (1st line)

$2(5)+5 = 15$ (2nd line)

Since z components are ~~not~~ equal, these two lines do not intersect.

#19 1^{st} plane has normal $\vec{n}_1 = \langle 1, 2, 3 \rangle$

2^{nd} plane has normal $\vec{n}_2 = \langle 1, -1, -1 \rangle$

$\therefore$ Direction vector for line of intersection is

$\vec{m} = \vec{n}_1 \times \vec{n}_2 = \langle 1, 4, -3 \rangle$

$\therefore$ line of intersection is $\vec{\ell}(t) = \langle 1, 4, -3 \rangle t + \langle 2, 0, -3 \rangle$

or $\begin{cases} x = t + 2 \\ y = 4t \\ z = -3t - 3 \end{cases}$

#20 $\vec{r}(t) = \langle 2 + \cos 3t, 3 - \sin 3t, 4t \rangle \quad 0 \le t \le \pi/4$

$$s = \int_0^t \|\vec{r}\,'(u)\| du = \int_0^t \left(9 \sin^2 3u + 9 \cos^2 3u + 16\right)^{1/2} du$$

$$= \int_0^t (9 + 16)^{1/2} du = 5t$$

$$\therefore \vec{r}(s) = \left\langle 2 + \cos \tfrac{3}{5}s,\ 3 - \sin \tfrac{3}{5}s,\ \tfrac{4}{5}s \right\rangle$$

#21 $\vec{F}(x, y) = \langle x^3 - y^3, xy^2 \rangle$

$C = \{\vec{r}(t) = (t^2, t^3) : -1 \le t \le 0\}$

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_{-1}^{0} \vec{F}(x(t), y(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_{-1}^{0} \langle t^6 - t^9, t^8 \rangle \cdot \langle 2t, 3t^2 \rangle\, dt$$

$$= \int_{-1}^{0} 2t(t^6 - t^9) + 3t^{10}\, dt$$

$$= \int_{-1}^{0} 2t^7 - 2t^{10} + 3t^{10}\, dt$$

$$= \int_{-1}^{0} 2t^7 + t^{10}\, dt = \left(\frac{2t^8}{8} + \frac{t^{11}}{11}\right)\Bigg|_{-1}^{0}$$

$$= -\frac{1}{4} + \frac{1}{11} = \frac{-7}{44}$$