Math-158 Calculus 3 Final Examination Spring 2007

Answer 10 questions. Each question is worth 20 points.

1.Find a rotation that eliminates the xy-term from the equation

$$x^2 - xy + y^2 = 2.$$

Rewrite the equation in terms of the new coordinate system and sketch the curve displaying both coordinate systems.

2. The position of a particle at time t is given by

$$\overrightarrow{r}(t) = e^{-t}\ \overrightarrow{i} + \cos t\ \overrightarrow{j} + \sin t\ \overrightarrow{k}.$$

(a) Show that the speed of the particle at time t is $\sqrt{e^{-2t}+1}$.
(b) Find the acceleration vector at time t.
(c) Show that the curvature $k(t)$ is given by $k(t) = \sqrt{\frac{1+2e^{-2t}}{(1+e^{-2t})^3}}$.

3. (a) Find parametric equations for the line of intersection of the planes $x + 2y - 3z + 5 = 0$ and $-2x + 3y + 7z + 2 = 0$.
(b) Find an equation for a sphere with center $(0, 1, 5)$ and which is tangential to the plane $3x + 6y - 2z - 5 = 0$.

4. Let

$$f(x, y, z) = \frac{xyz}{x^2 + y^4 + z^4}$$

(a) Show that as $(x, y, z) \to (0, 0, 0)$ along the curve $x = at,\ y = bt,\ z = ct$, $f(x, y, z) \to 0$.
(b) Find $\lim\limits_{(x,y,z)\to(0,0,0)} f(x, y, z)$ as $(x, y, z) \to (0, 0, 0)$ along the curve $x = t^2,\ y = t,\ z = t$.
(c) Does $\lim\limits_{(x,y,z)\to(0,0,0)} f(x, y, z)$ exist? Justify your answer.

5. Let $f(x, y, z) = x^3z^2 + y^2z + z + 1$ and $\overrightarrow{a} = \overrightarrow{i} + 2\ \overrightarrow{j} - 3\ \overrightarrow{k}$.
(a) Find the directional derivative of f in the direction of $\overrightarrow{a}$.
(b) What is the equation of the level surface of f that contains the point $(1, 1-1)$? Find an equation for the tangent plane and parametric equations for the normal line to this surface at the point $(1, 1, -1)$.

6. Determine and classify all stationary points of the function
$f(x,y) = xy - x^3 - y^3$.

7. Let $F(x,y,z) = xyz$. Use the method of Lagrange multipliers to find the maximum and minimum values of F on the sphere $x^2 + y^2 + z^2 - 1 = 0$.

8.(a) Use a double integral to find the volume of the solid under the plane $z = 2x + y$ and over the rectangle $R = \{(x,y) : 3 \le x \le 5,\ 1 \le y \le 2\}$.
(b) Evaluate the double integral

$$\iint_R y\,dA$$

where R is the region in the first quadrant and enclosed between the circle $x^2 + y^2 = 25$ and the line $x + y = 5$.

9. Reverse the order of integration in $\int_0^1 \int_{4x}^4 e^{-y^2}\,dy\,dx$ and then evaluate the integral.

10. Use polar coordinates to evaluate

$$\iint_R \frac{1}{1 + x^2 + y^2}\,dA,$$

where R is the sector in the first quadrant bounded by $y = 0$, $y = x$, and $x^2 + y^2 = 4$.

11. Find the area of the surface on the cylinder $y^2 + z^2 = 9$ which is above the rectangle $R = \{(x,y) : 0 \le x \le 2,\ -3 \le y \le 3\}$.

12. Compute the triple integral

$$\iiint_G xyz\,dV$$

where G is the solid in the first octant that is bounded by the parabolic cylinder $z = 2 - x^2$ and the planes $z = 0$, $y = 0$, and $y = x$.

13. Use the transformation $u = x - 2y$, $v = 2x + y$ to evaluate

$$\iint_R \frac{x - 2y}{2x + y}\,dA$$

where R is the region enclosed by the lines $x - 2y = 1$, $x - 2y = 4$, $2x + y = 1$, and $2x + y = 3$.

14. Evaluate the line integral

$$\int_C (3x + 2y)dx + (2x - y)dy$$

where C is the curve $x = y^3$ from $(0, 0)$ to $(1, 1)$.

15. Evaluate the line integral using Green's Theorem and check the answer by evaluating it directly.

$$\oint_C -y\,dx + xdy,$$

where C is the unit circle $x^2 + y^2 = 1$ oriented counterclockwise.