Differential Equations

Final Examination

December 13, 2005

There are 12 problems, but only 10 of these are required. Each problem is worth 20 pts. To earn the full grade you must show your work.

1. (20pts) Use Euler's method with the prescribed Δt to approximate the solution to the initial value problem in the given interval. Solve the problem by elementary methods and compare the approximate values of y with the correct value.

$$\frac{dy}{dt} = 2t - 3y; \quad y(0) = 2; \quad \Delta t = 0.1 \text{ and } 0 \leq t \leq 1.$$

2. (20pts) If $y(t) = 3e^{t^2}$ is known to be the solution to the initial value problem given by

$$\frac{dy}{dt} + p(t)y = 0, \quad y(0) = y_0,$$

what must the function $p(t)$ and the constant y_0 be?

3. (20pts) For each of the following initial value problems, find the general solution to the differential equation. Next, impose the initial condition to find the solution to the initial value problem.

a) $t\dfrac{dy}{dt} - 4y = 0, \quad y(1) = 1;$

b) $\dfrac{dy}{dt} - 2y = e^{3t}, \quad y(0) = 3;$

c) $2\dfrac{dy}{dt} + (\cos t)y = -3\cos t, \quad y(0) = -4.$

4. (20pts) Find the general solution to each of the following first-order differential equations.

a) $\dfrac{dy}{dt} = 3(1+t^2)y;$

b) $(t^2+4)\dfrac{dy}{dt} + 2ty = t^2(t^2+4);$

c) $(1+t^2)\dfrac{dy}{dt} + 2ty = 0.$

5. (20pts) Classify equilibrium points as sinks, sources, or nodes, of the first-order differential equation:

$$\frac{dy}{dt} = y^2 + \lambda y + 1,$$

where λ is a parameter.

6. (20pts) Let $P(t)$ denote the population of a colony. Assume that population is modeled by the logistic model,

$$\frac{dP}{dt} = (1 - P)P - \frac{2}{9}.$$

a) Determine all the equilibrium populations;

b) Draw the corresponding phase lines;

c) Use the phase lines to sketch the graphs of solutions for the differential equation;

d) Identify the equilibrium populations as sinks, sources, or nodes.

7. (20pts) Find the general solution to the linear system.

$$\frac{dx}{dt} = x - 3y, \quad \frac{dy}{dt} = x + 5y.$$

8. (20pts) For the system

$$\frac{d\mathbb{Y}}{dt} = \begin{pmatrix} 3 & -4 \\ 2 & -1 \end{pmatrix} \mathbb{Y},$$

a) Find the eigenvalues and the eigenvectors;

b) Find the general solution of the system;

c) Find the solution with the initial condition $\mathbb{Y}(0) = (1, 0)$;

d) Sketch the phase plane, including the solution with the given initial condition; and

e) Sketch the $x(t)$ – and $y(t)$-graphs of the solution with the given initial condition.

9. (20pts) Consider the nonhomogeneous second-order differential equation given by

$$(E) \qquad \frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = -5e^{-3t}.$$

a) Solve the equation (E);

b) Find the solution of (E) that satisfies $y(0) = 1$ and $y'(0) = -2$.

10. (20pts) Find the Laplace transform, if it exists, of each of the following functions.

a) $f(t) = e^{at}$.

b) $g(t) = \cos t$.

c) $h(t) = e^{t^2}$.

11. (20pts)

a) Find the Laplace inverse of $g(p) = \dfrac{1}{(p+1)(p^2+4)}$;

b) Solve the initial value problem

$$\frac{d^2y}{dt^2} + 4y = 2\cos(3t) \text{ with } y(0) = 1, \quad y'(0) = 0.$$

12. (20pts)

a) Define the value of the parameter ϑ corresponding to resonance in the equation

$$\frac{d^2y}{dt^2} + 2y = 2\cos(\vartheta t).$$

b) Find the resonance solution satisfying $y(0) = 0, \quad y'(0) = 1.$