Differential Equations

Final Examination

December 12, 2006

There are 13 problems, but only 10 of these are required. Each problem is worth 20 points. To earn the full grade you must show your work.

1. (20pts) For a population of about 100,000 bacteria in a petri dish, we decide to model population growth through the differential equation
$$\frac{d\mathbf{P}}{dt} = k\mathbf{P},$$
where k is the population growth.
Suppose, 2 days later; that the population has grown to about 150,000 bacteria.
   (a) Find the growth rate k;
   (b) estimate the bacteria population after 7 days.

2. (20pts) Consider the differential equation
$$\frac{dw}{dt} = w^3 + 7w^2 + 12w.$$
   (a) Determine all the equilibrium solutions;
   (b) classify those equilibrium solutions as sinks, sources, or nodes;
   (c) find intervals of increase and decrease;
   (d) Sketch the phase line.

3. (20pts) Solve the initial-value problem
$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 6y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

4. (20pts) Consider the linear system $\frac{d\mathbf{Y}}{dt} = A\mathbf{Y}$, where A is the 2×2 matrix given by
$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ -4 & m \end{pmatrix}.$$
   (a) For what values of m is $\mathbf{Y} = (0, 0)$ the only equilibrium solution?
   (b) For what values of m does more than one equilibrium solution exist? In this case, how many are there?

5. (20pts) Find the real solutions to the forced second-order differential equation

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 5y = \sin 2t.$$

6. (20pts) Consider the linear system $\frac{d\mathbf{Y}}{dt} = A\mathbf{Y}$, where A is the 2×2 matrix given by

$$\mathbf{A} = \begin{pmatrix} -1 & 2 \\ -3 & 1 \end{pmatrix}.$$

(a) Find its eigenvalues and eigenvectors;

(b) find the real solutions of this system;

(c) find the solution satisfying the initial condition $\mathbf{Y}(0) = (1, -1)$;

(d) Sketch the $x(t)$- and $y(t)$-graphs of the solution in (c).

7. (20pts) Use the Laplace transform to solve the initial-value problem

$$\frac{d^2y}{dt^2} + 4y = 4t + 8, \quad y(0) = 4, \quad y'(0) = -1.$$

8. (20pts) Consider the 2×2 matrix $\mathbf{A} = \begin{pmatrix} -3 & -m \\ m & 1 \end{pmatrix}$, where m is a parameter.

(a) Determine all values of m for which A has distinct real eigenvalues;

(b) determine all values of m for which A has distinct complex eigenvalues;

(c) Determine all values of m for which A has repeated eigenvalues. In this case, find the general solution to the linear system

$$\frac{d\mathbf{Y}}{dt} = A\mathbf{Y}.$$

9. (20pts) Use the Laplace transform to solve the initial value problem

$$\frac{dy}{dt} + 4y = 2 + 3t, \quad y(0) = 1.$$

10. (20pts) Consider the linear system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & m^2 \\ -1 & m \end{pmatrix}\mathbf{Y},$$

where $m \neq 0$ is a real number.

(a) Classify the origin as a spiral sink, or a spiral source, or a center;

(b) sketch the phase portrait for both $m = 1$ and $m = -1$.

11. (20pts) Find the inverse Laplace transform of the following functions

(a) $F(s) = \dfrac{2s^2 + 3s - 2}{s(s+1)(s-2)}$;

(b) $F(s) = \dfrac{2s+1}{(s-1)(s-2)}$.

12. (20pts) Find the solution to the linear system

$$\frac{d\mathbf{Y}(t)}{dt} = A\mathbf{Y}(t)$$

in each of the following cases

(a) $\mathbf{Y}(t) = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$, $\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix}$, and $\mathbf{Y}(0) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$;

(b) $\mathbf{Y}(t) = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$, $\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 5 \end{pmatrix}$, and $\mathbf{Y}(0) = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

13. (20pts) Find the general solution.

(a) $ty' + 2y = t^2$;

(b) $(t^2 + 4)y' + 2ty = t^2(t^2 + 4)$.