HOWARD UNIVERSITY
Differential Equations –Math 159
Final Examination
Wednesday, May 2, 2007

Answer any 9 problems and Problem 13 (required)
Each problem is worth 20 points. To earn the full grade you must show your work

1. (20pts) The field mouse population satisfies the differential equation

$$\frac{dP}{dt} = 0.5P - 450,$$

where t is measured in months.

(a) Find the time at which the population becomes extinct given $P(0) = 850$, and

(b) Determine the initial population P_0, which would become extinct in exactly one year.

2. (20pts) Consider the differential equation

$$\frac{dy}{dt} = (y^2 - \alpha)(y^2 - 4).$$

Determine the bifurcation value(s) and draw the phase lines corresponding to values of the parameter α that are less than, greater than and equal to the bifurcation value(s).

3. (20pts) Use the Laplace transform to solve the initial value problem

$$\frac{d^2y}{dt^2} + 4y = \sin 2t, \quad y(0) = 1, \quad y'(0) = 0.$$

4. (20pts) Consider the second-order differential equation

$$y'' + 6y' + 13y = e^{-4x}\cos 5x.$$

(a) Find a particular solution of this equation of the form $y_p(x) = e^{-4x}(A\cos 5x + B\sin 5x)$.

(b) Find the general solution.

5. (20pts) Perform Euler's method with the given step size $\Delta t = 0.25$ on the interval $0 \le t \le 1$ to approximate the solution of the initial value problem:

$$\frac{dy}{dt} = t - y^2, \quad y(0) = 1.$$

Your answer should include a table of approximating values of the dependent variable, and should also include a sketch of the approximate solution.

6. (20pts) Suppose a species of fish in a particular lake has a population $P = P(t)$ that is described by the logistic model

$$\frac{dP}{dt} = 0.3P\left(1 - \frac{P}{2500}\right),$$

where the time t is measured in years.

(a) Adjust the model above to account for the harvesting each year of one-fourth of the fish population.

(b) Use a phase line analysis to determine what the revised model in part (b) predicts for the long term behavior of the fish population if the initial population satisfies P(0) = 2500.

7. (20pts) Consider the nonautonomous differential equation

$$\frac{dy}{dt} = y^2 - 2yt + t^2 + y - t + 1.$$

(a) Use the u-substitution $u = y - t$ to transform the given differential equation in y to an autonomous differential equation in u.

(b) Sketch the phase line for the equation you obtained in part (a), and use it to sketch the graphs of solutions of the original equation.

8. (20pts) Consider the matrix $A = \begin{pmatrix} -\alpha & \alpha \\ -1 & 0 \end{pmatrix}$, where α is a parameter.

(a) Determine α so that the matrix A has distinct real eigenvalues.

(b) Determine α so that the matrix A has repeated eigenvalues.

(c) Determine α so that the matrix A has complex eigenvalues.

(d) Let $\alpha = 1$. Find real solutions to the linear system

$$\frac{d\mathbf{Y}}{dt} = A\mathbf{Y}.$$

9. (20pts) A harmonic oscillator is modeled by the second-order equation

$$9\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + y = 0.$$

(a) Write the corresponding first-order system of ordinary differential equations.

(b) Find the eigenvalues and corresponding eigenvectors of this (linear) system.

(c) Find the particular solution satisfying y(0)=1, v(0)=y'(0)=1.

(d) Classify the oscillation of the harmonic oscillator (as underdamped, overdamped, critically damped, or undamped) and when appropriate give the natural period.

(e) Sketch the $y(t)$–and $v(t)$–graphs of the particular solution found in part (c).

10. (20pts) Consider the linear system $\frac{d\mathbf{Y}}{dt} = A\mathbf{Y}$, where $\mathbf{Y} = \begin{pmatrix} x \\ y \end{pmatrix}$, and A is the 2×2 matrix given by

$$\mathbf{A} = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}.$$

(a) Find its eigenvalues and eigenvectors.

(b) Find the real solutions of this system.

(c) Find the solution satisfying the initial condition $\mathbf{Y}(0) = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$.

(d) Sketch the $x(t)$- and $y(t)$-graphs of the solution in part (c).

11. (20pts) Use the Laplace transform to solve the initial value problem

$$\frac{dy}{dt} + 4y = 2 + 3t, \quad y(0) = 1.$$

12. (20pts) Consider the linear system

$$\frac{d\mathbf{Y}(t)}{dt} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & -1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \mathbf{Y}(t), \quad \text{where} \quad \mathbf{Y}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}.$$

(a) Compute the eigenvalues.

(b) Compute the eigenvectors.

(c) Find the solutions to this linear system.

(d) Find the solution of this system satisfying $\mathbf{Y}(0) = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$.

13. (20pts) **(Required)**

(a) Find the general solution of the differential equation: $\frac{dy}{dt} = 2ty^2 + 3t^2y^2$.

(b) Solve the initial-value problem: $\frac{dy}{dt} - 2y = e^{2t}, \quad y(0) = 5$.

(c) Solve the initial-value problem: $\frac{dy}{dt} = \frac{2y}{t} + 2t^2, \quad y(-2) = 4$.

14. (20pts) Find the inverse Laplace transform of the function:

(a) $F(s) = \frac{4}{(s-3)^2} - \frac{1}{s^2 - 2s + 10}$; and

(b) $F(s) = \frac{2s+4}{(s-2)(s^2 - 4s + 8)}$.