SENIOR COMPREHENSIVE EXAM

SPRING 2009
April 4, 2009
Please do all problems! Each problem is worth 10 points.

1. Find $\frac{dy}{dx}$ for the following functions.
   a. $2x + xy = y^2$
   b. $y = \frac{\ln x}{e^{2x}}$
   c. $y = \int_1^{x^2} \sin(t^2) \, dt$
   d. $y = \tan(\cos(5x))$

2. Find the following antiderivatives.
   a. $\int \sin^3 x \cos^2 x \, dx$
   b. $\int x^2 e^x \, dx$
   c. $\int \frac{x^3 + 2x - 1}{x} \, dx$

3. Give an example of a function that is continuous at a point but not differentiable at that point. Prove your example is continuous but not differentiable at the point.

4. Determine if the following series converges or diverges.
   a. $\sum_{n=1}^{\infty} \frac{\sqrt{x}}{x^2 + 2x - 1}$
   b. $\sum_{n=0}^{\infty} \frac{e^n}{n!}$
   c. $\sum \sin \left( \frac{mt^2}{2n^2 + n - 1} \right)$

5. Prove or disprove that $AB = BA$, whenever $A$ and $B$ are $2x2$ matrices.

6. Find the inverse of

\[
\begin{bmatrix}
2 & 0 & 0 \\
0 & 1 & 1 \\
2 & 0 & 1
\end{bmatrix}
\]
7. State the following theorems or definitions. (Each part is worth 2 points.)
   (a) Mean Value Theorem
   (b) Group
   (c) Heine-Borel Theorem
   (d) Bolzano-Weierstrass Theorem
   (e) First Isomorphism Theorem for Groups

8. Let \( A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \), where \( a \) and \( b \) are any fixed real numbers. Use induction to show that

\[ \forall n \in \mathbb{N}, A^n = \begin{bmatrix} a^n & 0 \\ 0 & b^n \end{bmatrix}. \]